Derangements → How Many? \( D_n \)

There are \( n! \) ways to make a list of length \( n \) using elements \( 1, 2, \ldots, n \) without repetition. The list is a derangement if the number \( j \) does not occupy position \( j \) of the list for any \( j = 1, 2, \ldots, n \).

A good list is a derangement. A bad list is one in which one (or more) elements \( j \) of \( 1, 2, \ldots, n \) appears at position \( j \) of the list. We must count the number of bad lists and subtract from \( n! \).

There are \( n \) ways in which a list could be bad: 1 in position 1, 2 in position 2, etc. Define these sets:

\[
\begin{align*}
A_1 &= \text{lists with 1 in pos 1} \\
A_2 &= \text{lists with 2 in pos 2} \\
A_n &= \text{lists with } n \text{ in pos } n
\end{align*}
\]

\( |A_1| = 1 \) is in position 1. The other \( n-1 \) elements can be anywhere. There are \((n-1)\)! of these lists, \( |A_1| = (n-1)! \).

Similarly, \( |A_2| = |A_3| = \cdots = |A_n| = (n-1)! \).

\( \Rightarrow |A_1| + |A_2| + \cdots + |A_n| = n(n-1)! \) since there are \( n \) sets.

Next, consider \( |A_1 \cap A_2| \), 1 is in position 1, 2 is in position 2. The other \( n-2 \) elements can be anywhere. There are \((n-2)\)! of these lists, \( |A_1 \cap A_2| = (n-2)! \).

Similarly, \( |A_i \cap A_j| = (n-2)! \) if \( i \neq j \).

There are \( \binom{n}{2} \) intersections of pairs of sets

\( \Rightarrow |A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n| = \binom{n}{2}(n-2)! \).

Next, there are \( \binom{n}{3} \) intersections of triples.

\( \Rightarrow |A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| = \binom{n}{3}(n-3)! \).
Following the pattern:
There are \( \binom{n}{k} \) terms of form \(|A_1 \cap A_2 \cap \ldots \cap A_k|\). The cardinality of each is \((n-k)!\). Because \(k\) of the positions are fixed, and the remaining \(n-k\) can be anywhere.

Therefore, by Inclusion/Exclusion, we have:
\[
|A_1 \cup \ldots \cup A_n| = \left( \binom{n}{1} (n-1)! - \binom{n}{2} (n-2)! + \binom{n}{3} (n-3)! - \ldots \pm \binom{n}{n} (n-n)! \right) / n!
\]

Now subtract from \(n!\):
\[
D_n = n! - \left[ \binom{n}{1} (n-1)! - \binom{n}{2} (n-2)! + \ldots \pm \binom{n}{n} (n-n)! \right] = \left( \binom{n}{0} n! - \binom{n}{1} (n-1)! + \binom{n}{2} (n-2)! - \ldots \mp \binom{n}{n} (n-n)! \right)
\]

\[
D_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!
\]

Since \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\) we have
\[
D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \Rightarrow D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\]

\[
D_n = n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!} \right]
\]