Suppose that we have \( n \) distinguishable balls that we want to distribute into \( m \) indistinguishable boxes, with the condition that each box must contain at least one ball. Show that the number of ways to do this is

\[
\binom{n}{m}, \text{ where } \binom{n}{m} = S(n, m), \text{ the Stirling number of the 2nd kind}
\]

**Solution:**

Let \( \binom{n}{m} \) = the number of ways to distribute \( n \) distinguishable balls into \( m \) indistinguishable boxes, with the condition that each box must contain at least one ball.

The \( n \) distinguishable balls correspond to the integers \( 1, \cdots, n \) and the \( m \) indistinguishable boxes to \( m \) pairwise disjoint subsets in a partition of \( \{1, 2, \cdots, n\} \). We know that each partition of \( \{1, 2, \cdots, n\} \) into \( m \) disjoint sets gives rise to \( m! \) onto functions from \( \{1, 2, \cdots, n\} \) to \( \{1, \cdots, m\} \). Let \( S(n, m) \) be the number of partitions of \( \{1, 2, \cdots, n\} \) into \( m \) pairwise disjoint sets. So

\[
\text{onto}(n, m) = m! \cdot S(n, m)
\]

where \( \text{onto}(n, m) \) is the number of onto functions from a set with \( n \) elements to a set with \( m \) elements.

Hence,

\[
\binom{n}{m} = S(n, m)
\]

These numbers arise in counting the number of ways of dividing a set of \( n \) elements into \( k \) pairwise disjoint subsets.

Thus if \( A = \{1, 2, 3, 4\} \) and \( k = 2 \) we would get the following subsets:

- \( \{1, 2, 3\}, \{4\} \)
- \( \{2, 3, 4\}, \{1\} \)
- \( \{1, 3, 4\}, \{2\} \)
- \( \{1, 2, 4\}, \{3\} \)
- \( \{1, 2\}, \{3, 4\} \)
- \( \{1, 3\}, \{2, 4\} \)
- \( \{1, 4\}, \{2, 3\} \)

Let \( s_2(m, k) = \# \) partitions of a set of \( m \) elements into \( k \) pairwise disjoint subsets or the number of ways to partition a set of \( n \) objects into \( k \) groups. Thus \( s_2(4, 2) = 7 \) (seven ways to divide the set with 4 elements into 2 disjoint subsets).

Stirling numbers of the second kind \( (s_2) \) can be defined recursively by
Let us now show that the recurrence formula follows from the enumerative definition. Evidently, there is only one way to partition $n$ objects into 1 group (everything is in that group), and only one way to partition $n$ objects into $n$ groups (every object is a group all by itself). Proceeding recursively, a division of $n$ objects $\{a_1, a_2, \cdots, a_n\}$ into $k$ groups can be achieved by only one of two basic maneuvers:

* We could partition the first $n-1$ objects into $k$ groups, and then add object $a_n$ into one of those groups. There are $k \times s_2(n-1,k)$ ways to do this.

* We could partition the first $n-1$ objects into $k-1$ groups and then add object $a_n$ as a new 1 element group. This gives an additional $s_2(n-1, k-1)$ ways to create the desired partition.

Mathematica code to recursively defined the Stirling numbers of the second kind:

```mathematica
Clear[s2]
s2[n_Integer, 1] := 1
s2[n_Integer, k_Integer] := s2[n, 1] /; n == k
s2[n_Integer, k_Integer] := k * s2[n - 1, k] + s2[n - 1, k - 1] /; (1 \leq k \&\& k < n)
```

Here are some diagrams representing the different ways the sets can be partitioned: a line connects elements in the same subset, and a point represents a singleton subset.
If \( n = m \) or the number of balls equals the number of indistinguishable boxes, then the number of ways distributing the balls is the \( n^{th} \) Bell number.

**Example.** Determine the number of ways of distributing \( n \) distinct ( distinguishable) objects among \( m \) identical (indistinguishable) boxes, if a) each box must get at least one object; b) not every box need receive an object; and c) repeat a), if the boxes are distinguishable.

**Soln:**

a) This is directly from the definition of \( s_2 \): \( s_2(n, m) \)

b.) \( \sum_{j=1}^{m} s_2(n,j) = s_2(n,1) + s_2(n,2) + \ldots + s_2(n,m) \)

c. This is just the number of surjections from the set of objects to the set of boxes, or \( m! \cdot s_2(n,m) \)

**Distribution with conditions into Indistinguishable boxes**

Another type of distribution problem is one in which the boxes are NOT distinct. Suppose, for example, we purchase 7 toys for Angela, Betty, and Cathy and place them in 3 unlabeled boxes so one box has 3 toys, and the other 2 have 2 toys each. We assume that removing the labels makes the boxes indistinguishable. In this case, the distribution 2,4,7; 1,3; 5,6 of toys into unlabeled boxes is the same as the distribution of 2,4,7; 5,6; 1,3. What is important is which toys end up together, not which box they are in.

We saw in the preceding discussion that if the boxes are distinct, there are \( \frac{n!}{3! \cdot 2! \cdot 2!} = 210 \) ways of placing the toys into the boxes. If however, the boxes are unlabeled, the two boxes containing 2 toys each are considered...
identical. Since there are 2! ways of arranging these 2 boxes, we must divide \( \frac{7!}{3! \cdot 2!} \) by 2!. There are, therefore, 105 distributions of 7 toys into 3 unlabeled (that is, indistinguishable) boxes with 3 toys in one box and 2 toys in each of the other 2 boxes.

We now generalize the last example. Suppose we must distribute \( n \) objects into \( k \) identical boxes. Let \( r_i \) be the number of boxes that will contain \( i \) objects. In our example, \( r_0 = 0, r_1 = 0, r_2 = 2, r_3 = 1, r_4 = r_5 = r_6 = r_7 = 0 \). There are several things you should note about the \( r_i \)'s. First, since there are \( n \) objects, there will be \( n + 1 \) \( r_i \)'s: \( r_0, r_1, \ldots, r_n \). Second, since there are \( k \) identical boxes to be filled, \( \sum_{i=0}^{n} r_i = k \). This implies that at most \( k \) of the \( r_i \)'s are nonzero. In our example, we had 8 \( r_i \)'s, and clearly \( \sum_{i=0}^{7} r_i = 3 \), which is the number of boxes to be filled. Finally, since \( r_i \) boxes will contain \( i \) objects, and there are a total of \( n \) objects, \( \sum_{i=0}^{n} i \cdot r_i = n \). In our example,

\[
\sum_{i=0}^{n} i \cdot r_i = (0 \cdot 0) + (1 \cdot 0) + (2 \cdot 2) + (3 \cdot 1) + (4 \cdot 0) + (5 \cdot 0) + (6 \cdot 0) + (7 \cdot 0) = 7
\]

If we take a box containing \( i \) objects, these \( i \) objects can be arranged in \( i! \) ways. The \( r_i \) boxes containing \( i \) objects can be arranged in \( r_i! \) ways. For each \( i \), therefore, there are \( (i!)^{r_i} \cdot r_i! \) ways of arranging the objects and the boxes containing them (the boxes are indistinguishable). This leads to the following result.

**Theorem.** The number of ways to distribute \( n \) distinct objects into \( k \) identical boxes with \( r_0 \) empty boxes, \( r_1 \) boxes containing 1 objects, etc., \( r_2 \) boxes containing 2 objects, \( r_n \) boxes containing \( n \) objects is

\[
\frac{(n!)^{r_1} \cdot (2!)^{r_2} \cdots (n!)^{r_n}}{(1!)^{r_0} \cdot (2!)^{r_1} \cdots (n!)^{r_n}}
\]

**Example.** A school gives 15 awards at graduation. Although we do not know which students get awards, we've discovered that 2 students will get 4 awards each, three students will get two awards each and one student will get one award. How many award distributions are possible?

**Soln:** We must distribute \( n = 15 \) objects (the awards) among \( k = 6 \) indistinguishable boxes (the students receiving awards - since we do not know who gets how many awards, the students are indistinguishable to us). Since 2 boxes (students) get 4 awards each, three boxes (students) get two awards each, and one box (student) gets one award, \( r_4 = 2, r_2 = 3 \) and \( r_1 = 1 \). All other \( r_i \)'s are zero. The last formula therefore gives us the total number of possible distributions:

\[
\frac{15!}{(1!)^{r_1} \cdot (2!)^{r_2} \cdots (4!)^{r_4}} = 23648625
\]

**SPECIAL CASES TABLE**
<table>
<thead>
<tr>
<th>Case</th>
<th>Description</th>
<th># ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(n) distinct objects into (k) distinct boxes with (r_1) objects in box 1, (r_2) objects in box 2, etc., (r_1 + r_2 + \cdots + r_k = n)</td>
<td>(\frac{n!}{r_1! \cdot r_2! \cdot \cdots \cdot r_k!})</td>
</tr>
<tr>
<td>2</td>
<td>(n) distinct objects into (k) identical boxes with (r_0) empty boxes, (r_1) boxes containing 1 objects, etc., (r_2) boxes containing 2 objects, etc.</td>
<td>(\frac{n!}{(1!)^{r_1} \cdot (2!)^{r_2} \cdots (n!)^{r_n}})</td>
</tr>
</tbody>
</table>

Comments:
1. Formula 2 above is derived from formula (1) by recalling that a partition of a set \(A\) is a collection of pairwise disjoint subsets of \(A\) whose union is \(A\). In formula (3), if we interpret the \(n\) distinct objects as the elements of \(A\) and the boxes as subsets of \(A\), the formula gives the number of ways in which a set can be partitioned in \(k\) subsets, each subset having a given number of elements.
2. If we want the total number of partitions of a set with \(n\) elements into disjoint, non-empty subsets, the solution is given by the Bell numbers.

Here is the sequence of Bell numbers:

Computing Bell Numbers using the recursive definition:

Find Bell numbers using recursive definition; Mathematica code:

\[
b[0] = 1;
b[n_] := \sum_{k=0}^{n-1} b[k] \cdot \text{Binomial}[n-1,k]
b[3]
\]

\(5\)

\(\text{bellnos} = \text{Table}[b[n], \{n,0,12\}]\)
\(\{1,1,2,5,15,52,203,877,4140,21147,115975,678570,4213597\}\)

Interesting, the Bell numbers can also be computed by the following Mathematica code:

\(\text{In[4]:= expr = } \text{NestList}[\text{Factor[D[#1, }x] &, \text{Exp[Exp[}x-1]-1\text{]}}, \text{n}])\)

\(\text{Out[4]=}\)
\[
\{e^{-x} + e^{-x}x, e^{-x} + e^{-x}x + e^{-x}x^2, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5 + e^{-x}x^6, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5 + e^{-x}x^6 + e^{-x}x^7, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5 + e^{-x}x^6 + e^{-x}x^7 + e^{-x}x^8, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5 + e^{-x}x^6 + e^{-x}x^7 + e^{-x}x^8 + e^{-x}x^9, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5 + e^{-x}x^6 + e^{-x}x^7 + e^{-x}x^8 + e^{-x}x^9 + e^{-x}x^{10}, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5 + e^{-x}x^6 + e^{-x}x^7 + e^{-x}x^8 + e^{-x}x^9 + e^{-x}x^{10} + e^{-x}x^{11}, e^{-x} + e^{-x}x + e^{-x}x^2 + e^{-x}x^3 + e^{-x}x^4 + e^{-x}x^5 + e^{-x}x^6 + e^{-x}x^7 + e^{-x}x^8 + e^{-x}x^9 + e^{-x}x^{10} + e^{-x}x^{11} + e^{-x}x^{12}\}\]
Problems

8. In how many ways can a wrench, hammer, plunger and saw be distributed among Moe, Larry and Curly? 
\( 3^4 = 81 \)

9. Balls numbered from 1 to 12 are dropped from the top of a maze, flow through the maze and land in slot A, B, C or D. In how many ways can the balls land if 
   a. 5, 3, 1 and 3 balls fall into slots A, B, C and D.
   b. an equal number of balls fall into each slot.
   c. two slots are empty and an equal number of balls fall into the other two slots? (HINT: First choose the two slots that balls do fall into, then fill them each with half of the balls.)

10. Suppose in exercise 2, the balls are not labelled. How many ways can the balls land 
   a. in part a? (1)
   b. in part b? (1)
   c. in part c? \( \binom{\frac{4}{2}}{\frac{4}{2}} = 6 \)

11. 15 cups and 10 plates are piled up to be washed. In how many ways can the dishes be separated into 3 piles? 
\[ \binom{15+3-1}{2} \binom{10+3-1}{2} \div 3! = 1496 \]

12. In how many ways can 9 distinct objects be distributed into four distinct boxes so there are 4 objects in box 1, 2 objects in box 2, one object in box 3, and two objects in box 4? 
\[ \frac{9!}{4!\cdot2!\cdot1!\cdot2!} = 3780 \]

13. In how many ways can a 9-element set be partitioned into 3 2-element subsets and one 3-element subset?

14. In a holdup, four 4 robbers get 17 $100 bills, a necklace and a ring. In how many ways can their take be divided up? 
\[ 2^4 \cdot \binom{\frac{17+4-1}{3}}{\frac{4}{3}} = 18240 \]

15. The bank robbers of the last problem are caught and receive a total of ten years in prison. If each robber receives a term that is a nonnegative integer (it is possible some receive zero years, that is, were freed), how many prison term combinations are possible?

16. The bank robbers in the last exercise are Jesse, Billy, Butch and Sundance. In how many ways could Jesse, Billy, Butch and Sundance be sentenced? 
\[ \binom{\frac{10+4-1}{3}}{3} = 286 \]
17. In how many ways can we distribute 3 identical watches and 5 distinct radios among All, Cal, Hal and Sal?

We can distribute 3 identical watches to four distinct people in \( \binom{4+3-1}{3} = 20 \)

By formula \# , we can distribute 5 distinct radios to 4 distinct people in \( 4^5 = 1024 \) ways. By the multiplication rule, we can distribute 3 identical watches and 5 distinct radios among the 4 people in \( \binom{4+3-1}{3} \cdot 4^5 = 20480 \) ways.