### DEFINITIONS AND CONCEPTS

**Multiplying and Dividing Rational Expressions—continued**

To divide rational expressions, multiply the first by the reciprocal of the second.

\[
\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{AD}{BC}
\]

Then simplify, if possible.

**EXAMPLES**

Divide, and then simplify, if possible.

\[
\frac{x^2 + 4x + 3}{x^2 + 3x} \div \frac{3}{x} = \frac{x^2 + 4x + 3}{x^2 + 3x} \cdot \frac{x}{3}
\]

Multiply the first rational expression by the reciprocal of the second.

Multiply the numerators.

\[
\frac{x^2 + 4x + 3}{x^2 + 3x} \cdot \frac{x}{3}
\]

Multiply the denominators.

\[
\frac{(x + 1)(x + 3)}{x^2 + 3x} \cdot \frac{x}{3}
\]

Factor completely and then simplify.

\[
\frac{x + 1}{3}
\]

Multiply the remaining factors in the numerator.

Multiply the remaining factors in the denominator.

### REVIEW EXERCISES

Perform the operations and simplify when possible.

15. \[\frac{3x^2 y^2}{35} \div \frac{21x^2 y^2}{(xy)}\]

16. \[\frac{x^3 + 4x^2 + 4x}{x^2 - 6} \div \frac{9 - x^2}{x^2 + 5x + 6}\]

17. \[\frac{2a^2 - 5a - 3}{4a^2 - 36a} \div \frac{2a^2 + 5a + 2}{2a^2 + 5a - 3}\]

18. \[\frac{x^2 - 4t^2}{t} \div (t^2 + 2t)\]

19. \[\frac{h - 2}{h^3 + 4} \div \frac{4h}{h + 6}\]

20. \[\frac{m^2 + 3m + 9}{m^2 + 4m + 16} \div \frac{m^2 - 27}{m^2 + 5m - 6}\]

21. \[\frac{8m^2 + 6m - 9n^2}{2m^2 + 5mn + 3n^2} \div \frac{12m^2 + 7mn - 12n^2}{2m^2 + 8m - 12}\]

22. \[\frac{x^3 + 3x^2 + 2x}{2x^2 - 2x - 12} \div \frac{x^3 - 3x^2 - 4x}{3x^2 - 3x - x^2 + 3x + 2}\]

### DEFINITIONS AND CONCEPTS

**Adding and Subtracting Rational Expressions**

To add (or subtract) two rational expressions with like denominators, add (or subtract) the numerators and keep the common denominator.

**EXAMPLES**

Add:

\[
\frac{x^2 - 26}{x - 5} + \frac{1}{x - 5} = \frac{x^2 - 26 + 1}{x - 5}
\]

Add the numerators. Write the sum over the common denominator, \(x - 5\).

Combine like terms.

To simplify the result, factor the numerator and remove the factor common to the numerator and denominator.

The denominators of two rational expressions are given. Find the LCD:

\[
\left\{ \begin{array}{l}
x^2 - 8x + 16 = (x - 4)(x - 4) \\
9x - 36 = 3 \cdot 3 \cdot (x - 4)
\end{array} \right\}
\]

LCD = \(3 \cdot 3 \cdot (x - 4)(x - 4)\)

\[= 9(x - 4)^2\]
DEFINITIONS AND CONCEPTS

To add or subtract rational expressions with unlike denominators, find the LCD and express each rational expression with a denominator that is the LCD. Add (or subtract) the resulting fractions and simplify the result, if possible.

EXAMPLES

Subtract:

\[
\frac{2x}{x + 5} - \frac{1}{x} = \frac{2x - x}{x(x + 5)} = \frac{x + 5}{x(x + 5)}
\]

Build each rational expression to have the LCD of \(x(x + 5)\).

\[
= \frac{2x^2}{x(x + 5)} - \frac{x}{x(x + 5)}
\]

Multiply the numerators.

\[
= \frac{2x^2 - x}{x(x + 5)}
\]

Multiply the denominators.

\[
= \frac{2x^2 - x - 5}{x(x + 5)}
\]

Subtract the numerators. Write the difference over the common denominator.

The result does not simplify.

REVIEW EXERCISES

Perform the operations and simplify when possible.

23. \(\frac{5y}{x - y} - \frac{3}{x - y}\)
24. \(\frac{d^2 - a^2}{c^3 - d^3 + e^3 - a^3} + \frac{1}{d^3 - a^3}
25. \(\frac{4}{t - 3} + \frac{6}{3 - t} - \frac{2}{t - 3}
26. \(\frac{p + 3}{p^2 + 13p + 12} - \frac{2p + 4}{p^2 + 13p + 12} = \frac{1}{p - 12}

Perform the operations and simplify when possible.

31. \(\frac{9}{a + 1} - \frac{1}{a + 1}\)
32. \(\frac{5x}{14z^2} + \frac{y^2}{16z} = \frac{5x + 2y^2}{112z^2}
33. \(\frac{4x}{x - 4} - \frac{3}{x + 3} - \frac{3}{x + 4}\) + \frac{3}{12x + 20} = \frac{10 - 5x}{12x + 20}
34. \(\frac{2a + 4}{a + 2} - \frac{9}{a + 2}\)
35. \(\frac{y + 7}{y + 3} - \frac{y - 3}{y + 7} = \frac{14y + 5x}{16x + 5z}
36. \(\frac{4}{x - 6y} - \frac{10}{5x} = \frac{12x + 20}{(5 \times 5) - 5}
37. \(\frac{6}{a^2 - 9} - \frac{5}{a^2 - a - 6}\)
38. \(\frac{a}{a + 2} - \frac{3}{a^2 + 2a + 1} = \frac{2a - 3}{2a}

Finding the LCD:

15a^2b, 20ab^3 = \(10a^2b^3\)
27. \(ab^2 - ab, ab^2, b^2 - h = ab^2(h - 1)\)
29. \(x^2 - 4x - 5, x^2 - 25 = (x - 5)(x + 5)(x + 1)\)
30. \(m^2 - 4m + 4, m^3 - 8 = (m^2 - 2m + 4)(m - 2)^2\)

DEFINITIONS AND CONCEPTS

Complex fractions contain fractions in their numerators and/or their denominators.

Complex fractions: \(\frac{2}{t}, \frac{3}{m}, \frac{m + 4}{4t}, \text{ and } \frac{a^2 - b^2}{1 + a + b}\)
### Two methods are used to simplify complex fractions.

**Method 1:** Write the numerator and denominator as single fractions. Then divide the fractions and simplify.

This method works well when a complex fraction is written, or can be easily written, as a quotient of two single rational expressions.

<table>
<thead>
<tr>
<th>Simplify:</th>
<th>EXAMPLES</th>
</tr>
</thead>
</table>
| \[
\frac{4x^2}{y^3} \div \frac{14x}{y} = \frac{4x^2}{y^3} \cdot \frac{y}{14x}
\] | The main fraction bar of the complex fraction indicates division. |
| \[
= \frac{4x^2}{y^3} \cdot \frac{y}{14x}
\] | To divide rational expressions, multiply the first by the reciprocal of the second. |
| \[
= \frac{4x^2}{y^3} \cdot \frac{y}{14x}
\] | Multiply the numerators. |
| \[
= \frac{1}{y^3} \cdot \frac{4x}{14x}
\] | Multiply the denominators. |
| \[
= \frac{1}{y^3} \cdot \frac{4x}{14x}
\] | Factor the numerator and denominator. Then simplify by removing common factors of the numerator and denominator. |
| \[
= \frac{2x}{7y^3}
\] | Multiply the remaining factors in the numerator. |
| \[
= \frac{2x}{7y^3}
\] | Multiply the remaining factors in the denominator. |

**Method 2:** Determine the LCD of all the rational expressions in the complex fraction and multiply the complex fraction by 1 in the form \(\frac{\text{LCD}}{\text{LCD}}\).

This method works well when the complex fraction has sums and/or differences in the numerator or denominator.

<table>
<thead>
<tr>
<th>Simplify:</th>
<th>EXAMPLES</th>
</tr>
</thead>
</table>
| \[
\frac{1}{x - y} \div \frac{2}{2x} = \frac{1}{x - y} \cdot \frac{2}{2x}
\] | The LCD of all the rational expressions in the complex fraction is \(2x\). Multiply the complex fraction by 1 in the form \(\frac{2x}{2x}\). |
| \[
= \frac{1}{x - y} \cdot \frac{2}{2x}
\] | Multiply the numerators. |
| \[
= \frac{1}{x - y} \cdot \frac{2}{2x}
\] | Multiply the denominators. |
| \[
= \frac{1}{x} - \frac{2x - y \cdot 2x}{5}
\] | In the numerator, distribute the multiplication by \(2x\). |
| \[
= \frac{1}{x} - \frac{2x - y \cdot 2x}{5}
\] | In the denominator, perform the multiplication by \(2x\). |
| \[
= \frac{2 - 2xy}{5}
\] | In the numerator, perform each multiplication by \(2x\). |

### REVIEW EXERCISES

**Simplify each complex fraction.**

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>39.</td>
<td>(\frac{4a^2b^2}{9c^6} \div \frac{2b}{3a^2})</td>
</tr>
<tr>
<td>40.</td>
<td>(\frac{p^2 - 9}{6pt} \div \frac{p^2 + 5p + 6}{3pt})</td>
</tr>
<tr>
<td>41.</td>
<td>(\frac{1 + \frac{2}{a}}{b \cdot \frac{2b}{a}})</td>
</tr>
<tr>
<td>42.</td>
<td>(\frac{1 - \frac{2}{x}}{x^2} \div \frac{3}{x^2} \cdot \frac{4}{x^2} + \frac{3}{x^2})</td>
</tr>
<tr>
<td>43.</td>
<td>(\frac{4 + 1}{b + d} \div \frac{1 + \frac{1}{b + d}}{1 + \frac{1}{b + d}})</td>
</tr>
<tr>
<td>44.</td>
<td>(\frac{8}{r + 3} \div \frac{4 - \frac{2}{r^2 + r - 6}}{\frac{1}{r^2 + r - 6}})</td>
</tr>
</tbody>
</table>
### Dividing

#### DEFINITIONS AND CONCEPTS

**To divide monomials**, use the method for simplifying fractions or use the rules for exponents.

<table>
<thead>
<tr>
<th>Divide the monomials:</th>
<th>Divide the monomials:</th>
</tr>
</thead>
</table>
| \[
\frac{8p^2q}{20pq^3} = \frac{1}{4} \cdot \frac{1}{pq} = \frac{1}{5pq}\]
| \[
\frac{8p^2q}{20pq^3} = \frac{2p^2}{5pq} \]

*Keep each base and subtract the exponents.*

| Move \(q^{-2}\) to the denominator and change the sign of the exponent. |

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**To divide a polynomial by a monomial**, divide each term of the numerator by the denominator.

Long division can be used to **divide a polynomial by a polynomial** (other than a monomial). The long division method is a series of four steps that are repeated: Divide, multiply, subtract, and bring down the next term.

When the division has a remainder, write the answer in the form **Quotient + remainder** / **divisor**.

<table>
<thead>
<tr>
<th>Divide:</th>
<th>Divide:</th>
</tr>
</thead>
</table>
| \[
\frac{9c^3d^4 - 12c^3d^7}{27cd^6} = \frac{9c^3d^4 - 12c^3d^7}{27cd^6} = \frac{3c^3d^4}{9d^6} = \frac{c^3}{3d^2}
\]
| \[
6x^3 - x^2 + 6x + 5 \text{ by } 2x + 1.
\]

*Do each monomial division.*

The first division: \(\frac{6x^3}{2x} = 3x^2\).

The second division: \(\frac{-x^2}{2x} = -\frac{x}{2}\).

The third division: \(\frac{6x}{2x} = 3\).

The remainder is \(1\).

Thus \(6x^3 - x^2 + 6x + 5 = 3x^2 - 2x + 4 + \frac{1}{2x + 1}\).

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#### REVIEW EXERCISES

**Perform each division. Write answers using positive exponents.**

- \(256x^2 y^2 \div 512x^2 y^2 = \frac{1}{2}\)
- \((5x^3 y^2 z^3)^{10} + (10x^3 y^2 z^2)^{20} = \frac{1}{5}\)

**Perform each division.**

- \(x^3 + 2x^2 + 3x + 1 \div x + 1 = \frac{x^2 + 3x + 1}{x + 1}\)

**To divide** \(\frac{5x + x^3 + 3 + 3x^2}{x + 1}\), **write**: \(x + 1\) \(\text{to the power of} \ x^3 + 3x^2 + 5x + 3\)

**To divide** \(\frac{x^2 - 9}{x - 3}\), **write**: \(x - 3\) \(\text{to the power of} \ x^2 - 9\)

- \(51. \ b^2 + 5b + 20 \div b + 4 = \frac{1}{3} - 33v - 8y^2 + 3y^3 - 10\)
- \(52. \ b + 4\)
- \(53. \ x + 2) \div 8 = \frac{1}{2} - 2x + 4\)
- \(54. \ \text{Divide} \ (8m^2 - 18m - 9) \div 4m + 1 = \frac{3 \div 4m - 5 - 4}{1}\)
- \(55. \ (3a^2 - 2a^2 - 8) \div (a^2 + 5) = \frac{3a - 2 \div 15a^2 + 2}{a^2 + 5}\)
- \(56. \ \frac{m^8 + m^6 - 4m^4 + 5m^2 - 1}{m^8 + 3}\)

**Perform each division.**

- \(30x^2y^3 - 15x^2y - 10xy^2 \div -10xy = \frac{3x^2y + 3y - 1}{2} + y\)
REVIEW EXERCISES

Solve each equation, if possible.

67. \( \frac{4}{x} - \frac{1}{10} = \frac{7}{2x} \)
68. \( \frac{11}{t} = \frac{6}{t-7} \)
69. \( \frac{3}{y} - \frac{2}{y+1} = \frac{1}{2} \)
70. \( \frac{2}{3x+15} - \frac{1}{18} = \frac{1}{3x+12} \)
71. \( \frac{3}{x+2} = \frac{2}{2-x} + \frac{2}{x^2-4} \)
72. \( \frac{x+3}{x^2-7x+10} + \frac{2x^2+6}{x-2} = \frac{3x}{x-2} \)
73. \( \frac{5a}{a-3} - 7 = \frac{15}{a-3} \) No solution; 3 is extraneous

74. a. Simplify: \( \frac{10}{x^2-4x} + \frac{4}{x-4} = \frac{5}{x+y-41} \)
b. Solve: \( \frac{10}{x^2-4x} - \frac{4}{x} = \frac{5}{x-4} \)

Solve each formula for the indicated variable or expression.

75. \( H = \frac{2ab}{a+b} \) for \( b \)
76. \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) for \( y \)
77. \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \) for \( R \)
78. \( k = \frac{ma}{F} \) for \( F \)

DEFINITIONS AND CONCEPTS

Rate of Work: If a job can be completed in \( x \) units of time, the rate of work can be expressed as \( \frac{1}{x} \) of the job is completed per unit of time.

Shared-work problems:

Work completed = rate of work \( \cdot \) time worked

EXAMPLES

PRINTERS Working alone, a 300-A model printer can print a company’s payroll checks in 30 minutes. A 500-X model can print the same checks in 20 minutes. How long will it take if the printers work together to print the checks?

Analyze the problem Let \( x \) = the number of minutes it will take the printers, working together, to print the checks. Enter the data in a table.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Time (minutes)</th>
<th>Work Completed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 300-A</td>
<td>( \frac{1}{30} )</td>
<td>( \frac{x}{30} )</td>
</tr>
<tr>
<td>Model 500-X</td>
<td>( \frac{1}{20} )</td>
<td>( \frac{x}{20} )</td>
</tr>
</tbody>
</table>

Form an equation The part of the job done by the 300-A model plus the part of the job done by the 500-X model equals 1 job completed.

\[ \frac{x}{30} + \frac{x}{20} = 1 \]

Solve the equation

1. Multiply both sides by the LCD, 60.

\[ 60 \left( \frac{x}{30} + \frac{x}{20} \right) = 60(1) \]

2. Distribute the multiplication by 60.

\[ 60 \left( \frac{x}{30} \right) + 60 \left( \frac{x}{20} \right) = 60(1) \]

3. Perform each multiplication by 60.

\[ 2x + 3x = 60 \]

4. Combine like terms.

\[ 5x = 60 \]

5. Divide by 5.

\[ x = \frac{60}{5} = 12 \]

State the conclusion Working together, it will take the printers 12 minutes to print the checks.