Please note that the graphs in this handout are incomplete. They do not have the axes labeled with the name of the variable and they do not have any sense of scale on either axis. Still, even with these omissions, I think you will find this a helpful handout.
Transformations

To reduce the amount of work we must do when graphing functions or relations, we use transformations. Through the use of transformations we may make use of our previous work in graphing, extending our collection of graphs. Transformations reflect the natural relations among graphs, rigid motions; 1) rotation, 2) translation, and 3) change of scale (stretching or shrinking. These transformations (1 - 3) may be combined in any manner we choose, and they commute, which means the order in which we apply them doesn't matter. For example, rotating a graph and then translating it is the same as translating it first and then rotating it. It should be noted that rotating a graph $180^\circ$ produces the same effect as reflecting it through the origin. It will be convenient for our purposes to consider a reflection as a negative change of scale, not a rotation.

The following theorems on transformation of graphs are given. We discuss only translation and changes of scale. Theorems 1 and 2 are horizontal and vertical versions of translation respectively. Theorems 3 and 4 are similar versions of change of scale, including reflection. The similarity of statement between Theorems 1 and 2 and between Theorems 3 and 4 should be noted.

Transforming a graph by translating it horizontally and/or vertically is a rigid motion. A rigid motion treats a graph like a rigid piece of wire. Translating a graph horizontally moves it either left or right by a fixed amount. Similarly, vertical translation moves the graph either up or down by a fixed amount. Changing the scale of a graph stretches or shrinks it horizontally or vertically. A negative change of scale is a change of scale with a reflection as well, again either horizontally or vertically.

**Theorem 1: Horizontal Translation**

In an equation of a relation, replacing $x$ by $x - h$ translates the graph $|h|$ units horizontally.

1. If $h > 0$, the translation is to the right (positive $x$-direction).
2. If $h < 0$, the translation is to the left (negative $x$-direction).

**Theorem 2: Vertical Translation**

In an equation of a relation, replacing $y$ by $y - k$ translates the graph $|k|$ units vertically.

1. If $k > 0$, the translation is up (positive $y$-direction).
2. If $k < 0$, the translation is down (negative $y$-direction).

**Theorem 3: Horizontal Change of Scale**

In an equation of a relation, replacing $x$ by $\frac{x}{a}$ does the following to the graph:

1. If $|a| < 1$, the graph is shrunk horizontally.
2. If $|a| > 1$, the graph is stretched horizontally.
3. If $a < 0$, the graph is reflected horizontally as well.

**Theorem 4: Vertical Change of Scale**

In an equation of a relation, replacing $y$ by $\frac{y}{a}$ does the following to the graph:

1. If $|a| < 1$, the graph is shrunk vertically.
2. If $|a| > 1$, the graph is stretched vertically.
3. If $a < 0$, the graph is reflected vertically as well.
Examples

The graph on the left below is $y = |x|$. The graph on the right is $y = |x - 5|$. You can see how in accordance with Theorem 1, $h = +5$ and the graph has translated (shifted) 5 units to the right.

![Graph of $y = |x|$](image1)

![Graph of $y = |x - 5|$](image2)

The next graph (lower left) is $y - 5 = |x|$. (This is the same as $y = |x| + 5$.) From Theorem 2, we see that $k = +5$ and the graph has been translated (shifted) up 5 units. The graph on the lower right has both a vertical and a horizontal translation. It represents the equation $y - 7 = |x + 4|$. $h = -4$ and $k = 7$. The graph has been shifted 4 units left (left because $h < 0$) and 7 units up.

![Graph of $y - 5 = |x|$](image3)

![Graph of $y - 7 = |x + 4|$](image4)

We'll use the graph of $y = e^x$ to investigate the changes of scale described in Theorems 3 & 4. The first graph below (left) is $y = e^x$. Note that it crosses the y-axis at (0,1) and the x-axis is an asymptote. It also contains the points $(1,e)$ and $\left(-1, \frac{1}{e}\right)$. On the right is the graph $y = e^{x/3}$. The graph has not been shifted; it still crosses the y-axis at (0,1) and is still asymptotic to the x-axis. It has, however been stretched horizontally. Instead of containing the points $(1,e)$ and $\left(-1, \frac{1}{e}\right)$, it now contains $(3,e)$ and $\left(-3, \frac{1}{e}\right)$. 
The graph on the lower left is \( \frac{y}{3} = e^x \) (or \( y = 3e^x \)). It has not been shifted but it has been stretched vertically. Instead of containing the points \((1,e), \left(-1, \frac{1}{e}\right),\) and \((0,1)\) it now contains the points \((1, 3e), \left(-1, \frac{3}{e}\right),\) and \((0,3)\). Essentially, for a given x-coordinate the corresponding y-coordinate has been tripled. On the lower right is the graph of \( \frac{y}{-3} = e^x \) (or \( y = -3e^x \)). It is similar to \( \frac{y}{3} = e^x \) in that it has been stretched, but it has also been reflected about the x-axis. In comparison with the previous graph, y has been replaced by \( -y \).
All of the following graphs are drawn to the same scale.

\[ x^2 + y^2 = 36 \]
\[ x^2 + (y/2)^2 = 36 \text{ (a vertical stretch)} \]
\[ \left(\frac{x}{2}\right)^2 + y^2 = 36 \text{ (a horiz. stretch)} \]
\[ (2x)^2 + y^2 = 36 \text{ (shrunk horizontally)} \]
\[ x^2 + (2y)^2 = 36 \text{ (shrunk vertically)} \]
Take a look at the next 22 graphs and observe how they have been translated, stretched, shrunk, and/or reflected.

\[ y = \ln(x) \]

\[ y = \ln(-x) \]

\[ \frac{-y}{2} = \ln(x) \]

\[ y = x^3 \]

\[ y = \left(\frac{x}{2}\right)^3 \]
\[
\frac{y}{2} = x^3 \quad \text{(or } y = 2x^3) \quad y = (2x)^3
\]

\[
y + 2 = (x - 2)^3 \quad -y + 2 = (x - 2)^3
\]
\[
y = x^2
\]

\[
\frac{(y+4)}{1^2} = (x - 3)^2 \text{ (or } y = -4 + 2(x - 3)^2\text{)}
\]

\[
y = (x - 3)^2
\]

\[
y = (-x - 3)^2
\]
$y = \sin x$

$\frac{y}{3} = \sin x \text{ (or } y = 3\sin x)\

\frac{y}{\frac{1}{3}} = \sin x \text{ (or } y = \left(\frac{1}{3}\right)\sin x)$
\[ y = \sin x \]

\[ y = \sin 3x \]

\[ y = \sin \left( \frac{x}{3} \right) \]
$y = \sin 3x$

$y = \sin \left( 3 \left( x - \frac{\pi}{4} \right) \right)$

$y = \sin \left( 3x - \frac{\pi}{4} \right)$