Show that $8^n - 3^n$ is divisible by 5 for all natural numbers $n$.

First of all, what does it mean for a number to be “divisible by 5”? Well, if $A$ is “divisible by 5” then we know that there exists some natural number, $m$, such that $A = 5m$. That’s it.

Secondly, note that if we add two natural numbers together, the result is also a natural number. Now we can get on with our problem.

\[ P(n): \quad 8^n - 3^n \text{ is divisible by 5} \]

\[ P(1): \quad 8^1 - 3^1 = 8 - 3 = 5 \text{ which is certainly divisible by 5. } [5(1) = 5] \]

\[ P(k): \quad 8^k - 3^k = 5m, \text{ where } m \text{ is a natural number} \]

\[ P(k + 1): \quad 8^{k+1} - 3^{k+1} = 5\, \mathbb{Q}, \text{ where } \mathbb{Q} \text{ is a natural number} \]

\[
8^{k+1} - 3^{k+1} \quad \text{Multiply it out to see.}
\]
\[
= (8^k - 3^k)(8^1 + 3^1) - 3(8^k) + \{8(3^k)\}
\]

\[
8^k - 3^k = 5m \quad \text{from } P(k)
\]

\[
8^{k+1} - 3^{k+1} = 5m \cdot (8 + 3) - 3(8^k) + \{8(3^k)\}
\]

\[
8^k = 3^k + 5^k \quad \text{because } 8 = 3 + 5
\]

\[
8^{k+1} - 3^{k+1} = 5(11)m -3(8^k - 3^k) + 5(3^k)
\]

\[
8^{k+1} - 3^{k+1} = 55m - 3(5m) + 5(3^k)
\]

\[
8^{k+1} - 3^{k+1} = 40m + 5(3^k)
\]

\[
8^{k+1} - 3^{k+1} = 5(8m + 3^k)
\]

\[
8^{k+1} - 3^{k+1} = 5\, \mathbb{Q}, \text{ where } \mathbb{Q} \text{ is the natural number } (8m + 3^k)\]