Page 7 # 4. Show that a function is a solution of the equation \( y' + ay = 0 \), where \( a \) is a constant, if and only if it is a solution of the equation \( (e^{ax}y)' = 0 \). Hence show that the general solution of the equation is described by the formula \( y = ce^{-ax} \), where \( c \) is an arbitrary constant.

\[
\text{Step 1: Show } y' + ay = 0 \iff (ye^{ax})' = 0 \\
\text{Step 2: Show that the general solution to } y' + ay = 0 \text{ is } y = ce^{-ax}.
\]

\text{Step 1 requires two steps:}

\text{Step 1A: show } y' + ay = 0 \implies (ye^{ax})' = 0 \\
\text{Step 1B: show } (ye^{ax})' = 0 \implies y' + ay = 0

\textbf{Step 1A: Let } y = f(x). \text{ Then we know: } f'(x) + af(x) = 0 \text{ (because that is what is given)} \text{ Now:}

\[
(ye^{ax})' = e^{ax} f(x) + ae^{ax} f(x) \quad \text{differentiation} \\
= e^{ax} [f(x) + af(x)] \quad \text{factoring} \\
= e^{ax} (0) \quad \text{using the given info} \\
= 0
\]

So, we have shown \( y' + ay = 0 \) necessarily implies that \( (ye^{ax})' = 0 \)

\textbf{Step 1B:} Again, let \( y = f(x) \). Then we know that \( (e^{ax}f(x))' = 0 \). \text{ (given)} \text{ Now:}

\[
0 = (f(x)e^{ax})' \quad \text{using the given info} \\
= e^{ax} f(x) + ae^{ax} f(x) \quad \text{differentiation} \\
= e^{ax} [f(x) + af(x)] \quad \text{factoring} \\
= f(x) + af(x) \quad \text{dividing by } e^{ax} \text{ (which can not = 0)}
\]

We have shown that \( (ye^{ax})' = 0 \) necessarily implies that \( y' + ay = 0 \).

Since we have shown that each implies the other, we have shown
\( y' + ay = 0 \iff (ye^{ax})' = 0 \) and we are through with Step 1.

[Note that an implication of our result is that the general solution for \( y' + ay = 0 \) is identical to that for \( (ye^{ax})' = 0 \).]

\textbf{Step 2:} Show that \( y = ce^{-ax} \) is the general solution of \( y' + ay = 0 \).

Well, we know that the general solution of \( y' + ay = 0 \) is the same as the general solution of \( (ye^{ax})' = 0 \).

\[
(ye^{ax})' = 0 \quad \text{if and only if} \\
ye^{ax} = C \quad \text{or, solving for } y \\
y = Ce^{-ax} \quad \text{and we are done}
\]