Example: Page 159 #13:

Let \( A \) be an \( m \times n \) matrix and let \( T\mathbf{x} = A\mathbf{x} \). Show that the range of \( T \) is spanned by the column vectors of \( A \). (Use Theorem 2.4.)

\[
\begin{bmatrix}
x_1 \\
x_2 \\
\cdot \\
\cdot \\
x_n
\end{bmatrix}
\]

The range of \( T \) is \( A\mathbf{x} \). Let \( A = [a_1, a_2, a_3, \ldots, a_n] \) and \( \mathbf{x} = 
\begin{bmatrix}
x_1 \\
x_2 \\
\cdot \\
\cdot \\
x_n
\end{bmatrix}
\]

The range of \( T \) is \( A\mathbf{x} = x_1a_1 + x_2a_2 + \ldots + x_na_n \)

Since the range of \( T \), \( A\mathbf{x} \), can be written as a linear combination of the column vectors of \( A \), the range of \( T \) is spanned by the column vectors of \( A \).

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Let \( T\mathbf{x} = A\mathbf{x} \), where \( A \) is the given matrix. Find the dimension of, and a basis for, the range of \( T \). (Use Theorem 3.10 and the result of exercise #13.)

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
-2 & 5 & 1 & 4 \\
0 & 2 & 2 & 4
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 & 5 & 5 & 10 \\
0 & 2 & 2 & 4
\end{bmatrix} \rightarrow 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 & 5 & 0 & 0 \\
0 & 2 & 0 & 0
\end{bmatrix}
\]

Thus the dimension is 2 and a basis would be \((1, -2, 0)\) and \((0, 5, 2)\).

\[
\begin{bmatrix}
2 & 6 & -2 \\
1 & 3 & -1 \\
-1 & -3 & 2
\end{bmatrix} \rightarrow 
\begin{bmatrix}
2 & 0 & 0 \\
1 & 0 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\]

Thus the dimension is 2 and a basis would be \((2, 1, -1)\) and \((0, 0, 1)\).
c) \[
\begin{bmatrix}
1 & 1 & -3 \\
2 & -4 \\
0 & 1 \\
-3 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 \\
2 & -10 \\
2 & -5 \\
3 & -5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 \\
2 & 0 \\
1 & 0 \\
2 & 10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
1 & 1 & 0 \\
2 & 10
\end{bmatrix}
\]

The dimension is 2 and a basis is \((-1,0,1,2)\) and \((0,2,1,1)\)

\[
\begin{bmatrix}
1 & 1 & -1 & 2 \\
0 & 1 & 3 & -1 \\
-2 & 2 & -4 & 1 \\
1 & 0 & -2 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 3 & -1 \\
-2 & 4 & -6 & 5 \\
1 & -1 & -1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-2 & 4 & -18 & 9 \\
1 & -1 & 2 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & -1 & 0
\end{bmatrix}
\]

The dimension is 3 and a basis is: \((1,0,7,0)\), \((0,1,4,–1)\), and \((0,0,9,–1)\).

Now, the problem did not ask this, but suppose that for part c) we wanted to find an orthonormal basis, not just any old basis. (We could do this for parts a), b), and/or d) as well, but won’t.) We will call the old basis \(v_1\) and \(v_2\) and our new basis \(u_1\) and \(u_2\). 

\[v_1 = (-1,0,1,2)\]

\[||v_1|| = \sqrt{1 + 1 + 4} = \sqrt{6}\]

\[v_2 = (0,2,1,1)\]

\[u_1 = \frac{1}{\sqrt{6}} (-1,0,1,2)\]

\[u_2 = c \ (0,2,1,1) - \frac{1}{\sqrt{6}} (-1,0,1,2) \cdot (0,2,1,1) \frac{1}{\sqrt{6}} (-1,0,1,2)\]

\[= c \ (0,2,1,1) - \frac{3}{6} (-1,0,1,2) = \frac{c}{2} \ [ (1,4,1,0) ]\]

\[||u_2|| = 1 \Rightarrow c = \frac{\sqrt{2}}{3} \quad \text{and so} \quad u_2 = \frac{1}{3\sqrt{2}} (1,4,1,0)\]
If $V$ is the space of all continuous functions on $[0,1]$ and if $Tf = \int_0^1 f(x) \, dx$ for $f$ in $V$, show that $T$ is a linear transformation from $V$ into $\mathbb{R}^1$.

1. If $f(x)$ is continuous on $[0,1]$ then $\int_0^1 f(x) \, dx$ is defined and is a Real number and thus an element of $\mathbb{R}^1$. So, $T$ does map from $V$ into $\mathbb{R}^1$.

2. i) $T(cf) = \int_0^1 cf(x) \, dx = c \int_0^1 f(x) \, dx = cT(f)$

   ii) $T(f + g) = \int_0^1 (f + g)(x) \, dx$

   $= \int_0^1 (f(x) + g(x)) \, dx$

   $= \int_0^1 f(x) \, dx + \int_0^1 g(x) \, dx$

   $= Tf + Tg$

$T$ is a linear transformation.

From 1 we know that $T$ maps from $V$ into $\mathbb{R}^1$ and from 2 we know that $T$ is a linear transformation, thus we have shown that $T$ is a linear transformation from $V$ into $\mathbb{R}^1$. 