Let $U$ be a subspace of $\mathbb{R}^n$. Let $U^\perp$ be the set of all elements of $\mathbb{R}^n$ that are orthogonal to every element of $U$; that is, $v$ is in $U^\perp$ if $v \cdot u = 0$ for every element $u$ of $U$. Show that $U^\perp$ is a subspace of $\mathbb{R}^n$. (The subspace $U^\perp$ is called the orthogonal complement of $U$.)

$U^\perp$ is a subset of $\mathbb{R}^n$ (given), so we need to show that it ($U^\perp$) is:

i) not empty
ii) closed under addition
iii) closed under multiplication by a real number

i) $0 \cdot u = 0 \ \forall \ u \in U$, so $0 \in U^\perp$ thus $U^\perp$ is not empty.

ii) Let $v_1, v_2 \in U^\perp$, then $(v_1 + v_2) \cdot u = v_1 \cdot u + v_2 \cdot u = 0 + 0 = 0,$

so $(v_1 + v_2) \in U^\perp$ and $U^\perp$ is closed under addition.

iii) Let $v \in U^\perp$ and $a \in \mathbb{R}$, then $(a \ v) \cdot u = a(v \cdot u) = a(0) = 0,$

and thus $av \in U^\perp$ and $U^\perp$ is closed under multiplication by a real number.
Let $U$ be the subspace of $\mathbb{R}^3$ spanned by $(1,2,3)$ and $(2,-1,0)$.
Let $V$ be the subspace spanned by $(5,0,3)$ and $(0,5,6)$.
Is $V$ a subspace of $U$? Are $U$ and $V$ the same?

If we can find $a$, $b$, $c$, and $d \in \mathbb{R}$, \( a(1,2,3) + b(2,-1,0) \) and \( c(1,2,3) + d(2,-1,0) \),
then $V$ is a subspace of $U$. This gives:

\[
\begin{align*}
5 &= a + 2b \\
0 &= c + 2d \\
0 &= 2a - b \\
5 &= 2c - d \\
3 &= 3a \\
6 &= 3c
\end{align*}
\]
Which have solutions of 

\[ a = 1, \ b = 2 \quad \text{and} \quad c = 2, \ d = -1. \]

Thus, \( (5,0,3) = a(1,2,3) + b(2,-1,0) \) and \( (0,5,6) = c(1,2,3) + d(2,-1,0) \).
Since both \( (5,0,3) \) and \( (0,5,6) \) can be written as a linear combination of \( (1,2,3) \) and \( (2,-1,0) \),
\[ V \text{ is a subspace of } U. \]

Similarly, if we can find $\alpha$, $\beta$, $\gamma$, and $\delta \in \mathbb{R}$, \( \alpha(5,0,3) + \beta(0,5,6) \) and \( \gamma(5,0,3) + \delta(0,5,6) \),
then $U$ is a subspace of $V$. This gives:

\[
\begin{align*}
1 &= 5\alpha \\
2 &= 5\beta \\
2 &= 5\gamma \\
-1 &= 5\delta \\
3 &= 3\alpha + 6\beta \\
0 &= 3\gamma + 6\delta
\end{align*}
\]
Which have solutions of 

\[ \alpha = 1/5, \ \beta = 2/5, \ \gamma = 2/5, \ \delta = -1/5. \]

Thus, \( (1,2,3) = (1/5)(5,0,3) + (2/5)(0,5,6) \) and \( (2,-1,0) = (2/5)(5,0,3) - (1/5)(0,5,6) \).
Since both \( (1,2,3) \) and \( (2,-1,0) \) can be written as a linear combination of \( (5,0,3) \) and \( (0,5,6) \),
\[ U \text{ is a subspace of } V. \]

This, coupled with the result from the first part of the problem, allows us to conclude that $U = V$. 