Dirac Delta Function

Suppose that a force of magnitude $F$ acts on an object instantaneously. The goal is to describe mathematically a force that imparts an impulse of unit magnitude to an object at time $t$. (You can always multiply by a magnitude.) This is equivalent to being able to describe a metal bar being hit by a hammer.

The unit impulse function or Dirac delta function is defined in the following way.

1. $\delta_a(t) = 0, \ t \neq a$
2. $\int_{-\infty}^{\infty} \delta_a(t) \, dt = 1$

The Laplace transform $L(\delta_a(t)) = e^{-as}$

Find the answer to the following.

1. $y' - 2y = \delta_2(t), \ y(0) = 1$

2. $y' + 4y = 3\delta_1(t), \ y(0) = 0$

3. $y'' - 4y = \delta_3(t), \ y(0) = 0, y'(0) = 1$

4. $y'' + 16y = 4\cos 3t + \delta_a(t), \text{ where } a = \frac{\pi}{3} \text{ and } y(0) = 0, y'(0) = 0$

Answers:

1. $y = e^{2t} + u_2(t)[e^{2(t-2)}]$

2. $y = \frac{1}{3}(e^{5t} - e^{-t}) + u_3(t)[e^{5(t-3)}]$

3. $y = \frac{1}{2}[\sinh 2t + u_3(t)[\sinh 2(t-3)]]$

4. $y = \frac{4}{7}(\cos 3t - \cos 4t) + \frac{1}{4} u_a(t)[\sin 4(t-a)] \text{ where } a = \frac{\pi}{3}$
Convolution

When you have a transform of the form \( H(s) = F(s) \ G(s) \) or what appears to be a product in the world of Laplace, one option is to use convolution.

If \( H(s) = F(s) \ G(s) \) Then \( h(t) = f(t) * g(t) \) where the star represents convolution.

\[
\int_{0}^{t} f(t-\tau)g(\tau)d\tau
\]

Sometimes this is useful in finding the inverse transformation. You have several in the exercises.

Periodic Functions

The Laplace transform of a piecewise continuous periodic function \( f(t) \) with period \( p \) is

\[
L(f) = \frac{1}{1 - e^{-ps}} \int_{0}^{p} e^{-st}f(t)dt
\]

This is useful if the function pulse form repeats. There are problems on the exercise sheet.