Find the series solution valid around the ordinary point $x_0 = 1$ for this differential equation.

$$xy'' + y' + xy = 0$$

$$y = \sum_{n=0}^{\infty} a_n (x - 1)^n$$

$$y' = \sum_{n=1}^{\infty} a_n n (x - 1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n - 1) (x - 1)^{n-2}$$

$$xy'' + y' + xy = 0$$ gives:

$$x \sum_{n=2}^{\infty} a_n n(n - 1) (x - 1)^{n-2} + \sum_{n=1}^{\infty} a_n n (x - 1)^{n-1} + x \sum_{n=0}^{\infty} a_n (x - 1)^n = 0$$

Letting $x = 1 + (x - 1)$ gives us

$$\sum_{n=2}^{\infty} a_n n(n - 1) (x - 1)^{n-2} + (x - 1) \sum_{n=2}^{\infty} a_n n(n - 1) (x - 1)^{n-2} + \sum_{n=1}^{\infty} a_n n (x - 1)^{n-1} + \sum_{n=0}^{\infty} a_n (x - 1)^n + (x - 1) \sum_{n=0}^{\infty} a_n (x - 1)^n = 0$$

This cleans up, a bit, to:

$$\sum_{n=2}^{\infty} a_n n(n - 1) (x - 1)^{n-2} + \sum_{n=2}^{\infty} a_n n(n - 1) (x - 1)^{n-1} + \sum_{n=1}^{\infty} a_n n (x - 1)^{n-1} + \sum_{n=0}^{\infty} a_n (x - 1)^n + \sum_{n=0}^{\infty} a_n (x - 1)^{n+1} = 0$$
We want to start all of the series off at the same value of the exponent of \((x - 1)\), so let’s start them all at \((x - 1)^1\).

\[
2a_2 + \sum_{n=3}^{\infty} a_n n(n - 1) (x - 1)^{n-2} + \sum_{n=2}^{\infty} a_n n(n - 1) (x - 1)^{n-1} + a_1 + \sum_{n=2}^{\infty} a_n n (x - 1)^{n-1} + a_0 + \sum_{n=1}^{\infty} a_n (x - 1)^n + \sum_{n=0}^{\infty} a_n(x - 1)^{n+1} = 0
\]

Next, we want all of the series to start at the same value of the index. Since 2 of the 5 series already start at \(n = 2\), let’s have them all start there, but now I’m going to call it \(k = 2\).

So, for the first series, \(n = 3\) corresponds to \(j = 2\) and thus \(n = j + 1\). The second and third series are just fine as they are.

For the fourth series, \(n = 1\) corresponds to \(i = 2\), so \(n = i - 1\).

For the last series \(n = 0\) corresponds to \(m = 2\), so \(n = m - 2\).

Finally, letting \(j, i,\) and \(m = k\), we get:

\[
a_0 + a_1 + 2a_2 + \sum_{k=2}^{\infty} [a_{k+1} (k+1)(k) + a_k k(k-1) + a_k k + a_{k-1} + a_{k-2} ](x - 1)^{k-1} = 0
\]

Equating the coefficients of each individual power of \(x\) to zero gives:

\[
a_2 = \frac{-(a_0 + a_1)}{2} \quad \text{[From the coefficients of } x^0\text{.]} \\
a_{k+1} = \frac{-[a_k k^2 + a_{k-1} + a_{k-2}]}{k(k+1)} \quad \text{for } k \geq 2
\]

So, we have:

\[
a_0 = A \\
a_1 = B \\
a_2 = \frac{-(A + B)}{2} \\
a_3 = \frac{-[4a_2 + a_1 + a_0]}{2(3)} = \frac{A}{6} + \frac{B}{6}
\]
\[ a_4 = \frac{-[9a_3 + a_2 + a_1]}{4(3)} = -\frac{A}{12} - \frac{B}{6} \quad a_5 = \frac{-[16a_4 + a_3 + a_2]}{20} = \frac{A}{12} + \frac{3B}{20} \]

There does not appear to be a clear pattern, so we simply write:

\[ y = A[1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{12} + \frac{(x-1)^5}{12} + \ldots] + \]

\[ B[(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{6} + \frac{3(x-1)^5}{20} + \ldots] \]