Solution to Test #2

Problem 1. (15 points) Complete the table by filling the inverse functions.

<table>
<thead>
<tr>
<th>Function</th>
<th>Inverse function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) = -2x + 3 )</td>
<td>( h^{-1}(x) = \frac{1}{2}(-x + 3) ) OR ( h^{-1}(x) = \frac{1}{2}(3 - x) )</td>
</tr>
<tr>
<td>( f(x) = 2^x )</td>
<td>( f^{-1}(x) = \log_2 x )</td>
</tr>
<tr>
<td>( g(x) = \log_3 x )</td>
<td>( g^{-1}(x) = 3^x )</td>
</tr>
</tbody>
</table>

\[ x = -2y + 3 \quad \text{(switch then solve for } y) \Rightarrow 2y = -x + 3 \]
\[ \Rightarrow y = \frac{1}{2}(-x + 3) \quad \text{OR} \quad y = \frac{1}{2}(3 - x) \]

\[ x = 2^y \Rightarrow \quad \text{(switch then solve for } y) \Rightarrow y = \log_2 x \]

\[ x = \log_3 y \Rightarrow \quad \text{(switch then solve for } y) \Rightarrow y = 3^x \]

Problem 2. (18 points) Complete the table about the domain and range of the given functions.

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 3^{x-1} )</td>
<td>((-\infty, +\infty))</td>
<td>((0, +\infty))</td>
<td>(y = 0)</td>
</tr>
<tr>
<td>( h(x) = \ln x )</td>
<td>((0, +\infty))</td>
<td>((-\infty, +\infty))</td>
<td>(x = 0)</td>
</tr>
<tr>
<td>( g(x) = \log_3(x - 1) )</td>
<td>((1, +\infty))</td>
<td>((-\infty, +\infty))</td>
<td>(x = 1)</td>
</tr>
</tbody>
</table>

For the domain of function: \( g(x) = \log_3(x - 1) \)
\[ x - 1 > 0 \Rightarrow x > 1 \]

Problem 3. (12 points) Evaluate the following expressions:

(a) \[ \log_2 \sqrt{32} = \log_2 2^{\frac{5}{2}} = \frac{5}{2} \]

(b) \[ \log_8 4 = \log_8 8^{\frac{2}{3}} = \frac{2}{3} \]

Note: \[ 4 = \frac{8}{2} = 8 \cdot 2^{-1} = 8^{\frac{1}{3}} \cdot 8^{-\frac{1}{3}} = \left(8\right)^{\frac{2}{3}} \]

(c) \[ \log_2 56 - \log_2 7 = \log_2 \frac{56}{7} = \log_2 8 = \log_2 2^3 = 3 \]

Problem 4. (10 points) Write as a single logarithm:

\[ \log x - 2 \log(x^2 + 1) + \frac{1}{2} \log(3 - x^2) \]
\[ = \log \frac{x}{(x^2 + 1)^2} + \log \sqrt{3 - x^2} \]
\[ = \log \left( \frac{x}{(x^2 + 1)^2} \cdot \sqrt{3 - x^2} \right) = \log \frac{x\sqrt{3-x^2}}{(x^2+1)^2} \]

Problem 5 (20 points) Find the solutions to the equations:

(a) \[ 2^{x-1} = 10 \Rightarrow x - 1 = \log_2 10 \Rightarrow x = \log_2 10 + 1 \]
(b) \[ 5 \ln(x + 1) = 0 \implies \ln(x + 1) = 0 \implies x + 1 = e^0 = 1 \implies x = 0 \]
OR: \[ 5 \ln(x + 1) = 0 \implies \ln(x + 1)^5 = 0 \]
\[ (x + 1)^5 = e^0 = 1 \implies (x + 1)^5 = 1 \implies x + 1 = 1 \implies x = 0 \]
Mistakes: \[ 5 \ln(x + 1) = 0 \implies 5(x + 1) = 0 \]

(c) \[ 2^{x^3} = e^{2x} \implies \ln 2^{x^3} = \ln e^{2x} \implies (x + 3) \ln 2 = 2x \ln e = 2x \]
\[ (x + 3) \ln 2 = 2x \implies x \ln 2 + 3 \ln 2 = 2x \implies 2x - x \ln 2 = 3 \ln 2 \]
\[ \implies (2 - \ln 2)x = 3 \ln 2 \implies x = \frac{3 \ln 2}{2 - \ln 2} \]

(d) \[ \log_2(x + 2) + \log_2(x - 1) = 2 \implies \log_2[(x + 2)(x - 1)] = 2 \]
\[ (x + 2)(x - 1) = 2^2 = 4 \text{ (definition logarithm)} \implies x^2 + x - 2 - 4 = 0 \]
\[ x^2 + x - 6 = 0 \implies (x + 3)(x - 2) = 0 \]
\[ \implies x + 3 = 0 \quad \text{OR} \quad x - 2 = 0 \quad \text{(zero product property.)} \]
\[ \implies x = -3 \quad \text{OR} \quad x = 2 \]
Check:
For \( x = -3 \), \( \log_2(-3 + 2) + \log_2(-3 - 1) = \log_2(-1) + \log_2(-4) \) undefined. So, \( x = -3 \) is not a solution to \( \log_2(x + 2) + \log_2(x - 1) = 2 \)
For \( x = 2 \), \( \log_2(2 + 2) + \log_2(2 - 1) = \log_2 4 + \log_2 1 = \log_2 2^2 + 0 = 2 \). \( x = 2 \) is a solution to \( \log_2(x + 2) + \log_2(x - 1) = 2 \)
Solution to the equation \( \log_2(x + 2) + \log_2(x - 1) = 2 \) is: \( x = 2 \)

Problem 6. (10 points) The population of the world in 2000 was 6.1 billions, and the estimated relative growth was 1.4% per year. If the population continues to grow at this rate, when will it reach 122 billions?
(Use \( n(t) = n_0 e^{rt} \), \( \ln 2 = 0.693 15, \ln 5 = 1. 609 4 \))
\[ 122 = 6.1 e^{1.4\% t} \]
\[ e^{1.4\% t} = \frac{122}{6.1} = 20 \]
\[ e^{1.4\% t} =\frac{122}{6.1} = 20 \]
\[ 1.4\% \cdot t = \ln 20 \]
\[ 0.014 \cdot t = \ln 20 \]
\[ t = \frac{\ln 20}{0.014} = \frac{\ln 4 + \ln 5}{0.014} = \frac{2 \ln 2 + \ln 5}{0.014} = \frac{2 \cdot 0.693 15 + 1. 609 4}{0.014} = 213. 98 \]
214 years after 2000, there are 122 billions of people.
Problem 7. (6 points) The point \( P \) is on the unit circle. Find \( P(x,y) \) if the \( y \) coordinate is \( -\frac{\sqrt{3}}{2} \) and the \( x \) coordinate is positive.
Problem 8. (9 points) Suppose that the terminal point determined by \( t \) is the point \((-\frac{3}{5}, \frac{4}{5})\) on the unit circle. Please mark the terminal point determined by the following

(A) \( \pi - t \)  (B) \( \pi + t \)  (C) \(-t\)
\((-\frac{\sqrt{3}}{2})\)^2 + x^2 = 1, x > 0 \Rightarrow \frac{3}{4} + x^2 = 1 \Rightarrow x^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow x = \frac{1}{2} \text{ since } x > 0

P(x,y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)

Problem 9 (4 points) Fill out the domain and range of the functions:
\(f(x) = \log_2(x^2 + 1)\)
Math 180

Domain Range Asymptote

\[ f(x) = \log_2(x^2 + 1) \quad (-\infty, +\infty) \quad (-\infty, +\infty) \quad \text{None} \]

\[ g(x) = \log_2(x^2 - 1) \quad (-\infty, -1) \cup (1, \infty) \quad (-\infty, +\infty) \quad x = -1 \text{ and } x = 1 \]

Note: \( x^2 + 1 > 0 \) is true for any real number \( x \).

As for \( g(x) = \log_2(x^2 - 1) \), we must satisfy:

\[ x^2 - 1 > 0 \Rightarrow (x - 1)(x + 1) > 0 \Rightarrow x < -1 \text{ OR } x > 1 \]

Problem 10 (16 points) Use the given unit circle to complete the table for the values of the given functions.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \sin t )</th>
<th>( \cos t )</th>
<th>( \tan t )</th>
<th>( \cot t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>-1</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
</tr>
</tbody>
</table>

Problem 11. (10 points) The 1995 earthquake in Kobe, Japan is 7.2 – magnitude on the Richter scale. The recent earthquake in Asia is 7.6 – magnitude on the Richter scale. How many times more intense was the recent earthquake than the Kobe earthquake?

Solution:

7.2 – magnitude on the Richter scale \( \Rightarrow 7.2 = \log \frac{I_{\text{Japan}}}{S} \Rightarrow \frac{I_{\text{Japan}}}{S} = 10^{7.2} \Rightarrow I_{\text{Japan}} = 10^{7.2} \cdot S \)

7.6 – magnitude on the Richter scale \( \Rightarrow 7.6 = \log \frac{I_{\text{Asia}}}{S} \Rightarrow \frac{I_{\text{Asia}}}{S} = 10^{7.6} \Rightarrow I_{\text{Asia}} = 10^{7.6} \cdot S \)

\[ \frac{I_{\text{Asia}}}{I_{\text{Japan}}} = \frac{10^{7.6} \cdot S}{10^{7.2} \cdot S} = 10^{0.4} = 10^{\frac{2}{5}} = 2.5119 \]

The recent earthquake is about 2.5 more intense than the Kobe earthquake in 1995.