2.3 Laws of Limits:

Premises: \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist. Then the following hold.

**Theorem 1.** The limit of a sum is the sum of the limits:

\[
\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

2. The limit of a difference is the difference of the limits:

\[
\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)
\]

3. The limit of a constant times a function is the constant times the limit of the function:

\[
\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)
\]

4. The limit of a product is the product of the limits:

\[
\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)
\]

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if} \lim_{x \to a} g(x) \neq 0
\]

The following laws are immediate consequences of repeated applications of the preceding laws and the definition of limit.

6. The limit of a power of a function is the power of the limit of the function:

\[
\lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n \quad \text{where} \ n \ \text{is a positive integer}
\]

7. The limit of a constant function is the constant:

\[
\lim_{x \to a} c = c
\]

8. The limit of \( x \) as \( x \) approaches \( a \) is \( a \):

\[
\lim_{x \to a} x = a
\]

9. The limit of \( x^n \) as \( x \) approaches \( a \) is \( a^n \):

\[
\lim_{x \to a} x^n = a^n \quad \text{where} \ n \ \text{is a positive integer}
\]

10. The limit of the \( n \)-th root of \( x \) as \( x \) approaches \( a \) is the \( n \)-th root of \( a \):

\[
\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where} \ n \ \text{is a positive integer}
\]

(If \( n \) is even, assume that \( a > 0 \).)

11. The limit of the \( n \)-th root of \( f(x) \) is the \( n \)-th root of the limit of \( f(x) \):

\[
\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad \text{where} \ n \ \text{is a positive integer}\]
(If $n$ is even, assume that $\lim_{x \to a} f(x) > 0$.)

12. If $f$ is a polynomial or a rational function and $a$ is in the domain of $f$, then

$$\lim_{x \to a} f(x) = f(a)$$

**Limit Inequality**

**Theorem** If $f(x) \leq g(x)$ for all $x$ in an open interval that contains $a$ (except possibly at $a$) and the limits of $f$ and $g$ both exist as $x$ approaches $a$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

**Squeeze Theorem**

The Squeeze Theorem is sometimes called the Sandwich Theorem or the Pinching Theorem. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near $a$, and if $f$ and $h$ have the same limit $L$ at $a$, then $g$ is forced to have the same limit $L$ at $a$.

**Theorem** If $f(x) \leq g(x) \leq h(x)$ for all $x$ in an open interval that contains $a$ (except possibly at $a$) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

**Example** The plot below illustrates the squeeze theorem with the limit

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0$$

$$-|x| \leq x \sin \frac{1}{x} \leq |x|$$
\[ f(x) = \sqrt{x^3 + x^2 \sin \frac{x}{x}} \]

\[ g(x) = \sin \frac{1}{x} \]

\[ \lim_{x \to \infty} \frac{2x^2 + 3x - 1}{4x^2 + 2x + 5} = \lim_{x \to \infty} \frac{(2x^2 + 3x - 1) - \frac{1}{x^2}}{4x^2 - \frac{2x + 5}{x^2}} = \lim_{x \to \infty} \frac{2x^2 - \frac{1}{x^2} + 3x - \frac{1}{x} - 1 + \frac{1}{x}}{4x^2 - \frac{2x}{x^2} + 2x + \frac{5}{x^2}} \]

\[ = \lim_{x \to \infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{4 + \frac{2}{x} + \frac{5}{x^2}} = \frac{2}{4} = \frac{1}{2} \]

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(a) \( \lim_{x \to 2} f(x) = 2 \) and \( \lim_{x \to 2} f(x) = 2 \)

Thus, \( \lim_{x \to 2} f(x) = 2 \) according to Thm 1 on page 109.

\( \lim_{x \to 0} g(x) = 0 \)
By Limit Law #1:
\[ \lim_{x \to 2} [f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 2 + 0 = 2 \]
(b) \[ \lim_{x \to 1} f(x) = 1 \]
But \[ \lim_{x \to 1} g(x) = 1 \neq \lim_{x \to 1} g(x) = 2 \]
Therefore, \( \lim_{x \to 1} [f(x) + g(x)] \) does not exist.

(c) Since \[ \lim_{x \to 0} f(x) = 0 \text{ and } \lim_{x \to 0} g(x) = 1.5 \]
By Limit Law #4,
\[ \lim_{x \to 0} [f(x)g(x)] = \lim_{x \to 0} f(x) \cdot \lim_{x \to 0} g(x) 
= 0 \cdot 1.5 = 0 \]

(d) Since \[ \lim_{x \to -1} f(x) = -1 \text{ and } \lim_{x \to -1} g(x) = 0 \]

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\[ \lim_{x \to \infty} \frac{3x+5}{x-4} \]
\[ \lim_{x \to \infty} \frac{x^2+2}{x^3+x^2-1} \]

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#19. \[ \lim_{x \to 0} \frac{\sqrt{x^4-2}}{x} = 0.25 \]
Table:

<table>
<thead>
<tr>
<th>x</th>
<th>( \frac{\sqrt{x^4-2}}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>( \frac{\sqrt{0.01^4-2}}{0.01} ) = 0.24984</td>
</tr>
<tr>
<td>-0.01</td>
<td>( \frac{-\sqrt{-0.01^4-2}}{-0.01} ) = 0.25016</td>
</tr>
<tr>
<td>0.001</td>
<td>( \frac{\sqrt{0.001^4-2}}{0.001} ) = 0.24998</td>
</tr>
<tr>
<td>-0.001</td>
<td>( \frac{-\sqrt{-0.001^4-2}}{-0.001} ) = 0.25002</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} \frac{\sqrt{x^4-2}}{x} = \frac{\sqrt{x^4-2}}{x} \cdot \frac{\sqrt{x^4+2}}{\sqrt{x^4+2}} = \frac{(\sqrt{x^4})^2-2}{x(\sqrt{x^4+2})} = \frac{x^4-4}{x(\sqrt{x^4+2})} \]

\[ = \frac{x}{x(\sqrt{x^4+2})} = \frac{1}{\sqrt{x^4+2}} \]
\[ \lim_{x \to 0} \frac{\sqrt{x^4-2}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x^4+2}} = \frac{1}{\sqrt{2^2}} = \frac{1}{4} = 0.25 \]

Limit Laws:
Premise: \( c \) is a constant, the limits
\[ \lim_{x \to a} f(x) \quad \lim_{x \to a} g(x) \text{ exist.} \]

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#2. Check the premises:
a. Do the limits: \( \lim_{x \to 2} f(x) \quad \lim_{x \to 2} g(x) \text{ exist} \)
\( \lim_{x \to 2} f(x) = 2 \quad (f(2) = 1) \quad \lim_{x \to 2} g(x) = 0 \)
\( \lim_{x \to 2} [f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 2 + 0 = 2 \)

b. Do the limits: \( \lim_{x \to 1} f(x) \quad \lim_{x \to 1} g(x) \) exist
\( \lim_{x \to 1} f(x) = 1 \quad \lim_{x \to 1} g(x) \) does not exist.
\( \lim_{x \to 1} g(x) = 2 \neq \lim_{x \to 1} g(x) = 1 \)

Can not use \( \lim_{x \to 1} [f(x) + g(x)] = \lim_{x \to 1} (x) + \lim_{x \to 1} g(x) \)

Note: \( f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x} \)
\( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} \left[ \sqrt{x + 1} - \sqrt{x} \right] \)
\( = \lim_{x \to a} \left( \sqrt{x + 1} - \sqrt{x} \right) \cdot \frac{\left( \sqrt{x + 1} + \sqrt{x} \right)}{\left( \sqrt{x + 1} + \sqrt{x} \right)} \)
\( = \lim_{x \to \infty} \frac{\left( \sqrt{x + 1} \right)^2 - \left( \sqrt{x} \right)^2}{\left( \sqrt{x + 1} + \sqrt{x} \right)} = \lim_{x \to \infty} \frac{1}{\sqrt{x + 1} + \sqrt{x}} = 0 \)

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\( \lim_{x \to a} f(x) \quad \lim_{x \to a} g(x) \) exist.

3. \( \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) \)
\( \lim_{x \to a} c = c \)
\( \lim_{x \to a} x = a \)
\( \lim_{x \to a} x^n = a^n \)

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Theorem 2. If \( f(x) \leq g(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and the limit of \( f \) and \( g \) both exist as \( x \to a \), then
\( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \)

Premises: 1. \( f(x) \leq g(x) \)
2. \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist

Remark: \( f \) and \( g \) may/maynot be defined at \( a \).

Conclusion: \( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \)

Remark: Is it true?

Premises: 1. \( f(x) < g(x) \)
2. \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist

Remark: \( f \) and \( g \) may/maynot be defined at \( a \).

Conclusion: \( \lim_{x \to a} f(x) < \lim_{x \to a} g(x) \)

Counter example: \( f(x) = \frac{1}{x^2}, \quad g(x) = \frac{1}{x} \quad x > 1 \)

It is True:

Premises: 1. \( f(x) < g(x) \)
2. \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist

Remark: \( f \) and \( g \) may/maynot be defined at \( a \).

Conclusion: \( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \)

**Example** Show \( \lim_{x \to a} x^2 \sin \frac{1}{x} = 0 \).
Proof: \[ -1 \leq \sin \frac{1}{x} \leq 1 \]
\[ -1 \cdot x^2 \leq x^2 \sin \frac{1}{x} \leq 1 \cdot x^2 \]
\[ -x^2 \leq x^2 \sin \frac{1}{x} \leq x^2 \]
\[ \lim_{x \to 0} (-x^2) = \lim_{x \to 0} (x^2) = 0 \]
By The Squeeze Theorem, \( \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 \).

\[ f(x) = x^2 \sin \frac{1}{x} \]