2.3 Laws of Limits:
Premises: \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist. Then the following hold.

**Theorem 1.** The limit of a sum is the sum of the limits:

\[
\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

2. The limit of a difference is the difference of the limits:

\[
\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)
\]

3. The limit of a constant times a function is the constant times the limit of the function:

\[
\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)
\]

4. The limit of a product is the product of the limits:

\[
\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)
\]

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if} \lim_{x \to a} g(x) \neq 0
\]

The following laws are immediate consequences of repeated applications of the preceding laws and the definition of limit.

6. The limit of a power of a function is the power of the limit of the function:

\[
\lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n \quad \text{where } n \text{ is a positive integer}
\]

7. The limit of a constant function is the constant:

\[
\lim_{x \to a} c = c
\]

8. The limit of \( x \) as \( x \) approaches \( a \) is \( a \):

\[
\lim_{x \to a} x = a
\]

9. The limit of \( x^n \) as \( x \) approaches \( a \) is \( a^n \):

\[
\lim_{x \to a} x^n = a^n \quad \text{where } n \text{ is a positive integer}
\]

10. The limit of the \( n \)-th root of \( x \) as \( x \) approaches \( a \) is the \( n \)-th root of \( a \):

\[
\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where } n \text{ is a positive integer}
\]

(If \( n \) is even, assume that \( a > 0 \).)

11. The limit of the \( n \)-th root of \( f(x) \) is the \( n \)-th root of the limit of \( f(x) \):

\[
\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad \text{where } n \text{ is a positive integer}
(If \( n \) is even, assume that \( \lim_{x \to a} f(x) > 0. \))

12. If \( f \) is a polynomial or a rational function and \( a \) is in the domain of \( f \), then
   \[
   \lim_{x \to a} f(x) = f(a)
   \]

Limit Inequality

**Theorem** If \( f(x) \leq g(x) \) for all \( x \) in an open interval that contains \( a \) (except possibly at \( a \)) and the limits of \( f \) and \( g \) both exist as \( x \) approaches \( a \), then
   \[
   \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)
   \]

Squeeze Theorem

The Squeeze Theorem is sometimes called the Sandwich Theorem or the Pinching Theorem. It says that if \( g(x) \) is squeezed between \( f(x) \) and \( h(x) \) near \( a \), and if \( f \) and \( h \) have the same limit \( L \) at \( a \), then \( g \) is forced to have the same limit \( L \) at \( a \).

**Theorem** If \( f(x) \leq g(x) \leq h(x) \) for all \( x \) in an open interval that contains \( a \) (except possibly at \( a \)) and
   \[
   \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L
   \]
   then
   \[
   \lim_{x \to a} g(x) = L
   \]

Example The plot below illustrates the squeeze theorem with the limit
   \[
   \lim_{x \to 0} x \sin \frac{1}{x} = 0
   \]

\[-|x| \leq x \sin \frac{1}{x} \leq |x|\]
Analysis: \( f(x) = \sqrt{x^3 + x^2 \sin \frac{x}{x}} \)

We want to find \( g(x) \) and \( h(x) \) such that: \( g(x) \leq |f(x)| \leq h(x) \)

\[
|f(x)| = |\sqrt{x^3 + x^2 \sin \frac{x}{x}}| = |\sqrt{x^3 + x^2}| \cdot |\sin \frac{x}{x}|
\]

\[
\leq |\sqrt{x^3 + x^2}| \cdot 1 = |\sqrt{x^3 + x^2}| \quad \text{(since } |\sin \frac{x}{x}| \leq 1) \]

\[
\leq \sqrt{x^2 + x^2} = \sqrt{2x^2} = \sqrt{2} \sqrt{x^2} = \sqrt{2} |x| \quad \text{(Let } |x| \leq 1, x^3 \leq x^2) \]

\[
\frac{|f(x)|}{\sqrt{2} |x|} = \frac{|\sqrt{x^3 + x^2 \sin \frac{x}{x}}|}{\sqrt{2} |x|} \leq \frac{\sqrt{2} |x|}{\sqrt{2} |x|} = 1
\]

\[
g(x) \quad h(x)
\]

\[
g(x) = \sin \frac{1}{x}
\]
(a) \( \lim_{x \to 2^+} f(x) = 2 \) and \( \lim_{x \to 2^-} f(x) = 2 \)
Thus, \( \lim_{x \to 2} f(x) = 2 \) according to Thm 1 on page 109.
\( \lim_{x \to 2} g(x) = 0 \)
By Limit Law #1:
\( \lim_{x \to 2} [f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 2 + 0 = 2 \)

(b) \( \lim_{x \to 1^+} f(x) = 1 \)
But \( \lim_{x \to 1^-} g(x) = 1 \neq \lim_{x \to 1^-} g(x) = 2 \)
Therefore, \( \lim_{x \to 1} g(x) \) does not exist.
\( \lim_{x \to 1} [f(x) + g(x)] \) does not exist.

(c) Since \( \lim_{x \to 0^-} f(x) = 0 \) and \( \lim_{x \to 0} g(x) = 1.5 \)
By Limit Law #4,
\( \lim_{x \to 0} [f(x)g(x)] = \lim_{x \to 0} f(x) \cdot \lim_{x \to 0} g(x) \)
\( = 0 \cdot 1.5 = 0 \)

(d) Since \( \lim_{x \to 1^-} f(x) = -1 \) and \( \lim_{x \to 1^-} g(x) = 0 \)

Page 112.
\( \lim_{x \to \infty} \frac{2x^2 + 3x - 1}{4x^2 + 2x + 5} = \lim_{x \to \infty} \frac{(2x^2 + 3x - 1) \cdot \frac{1}{x^2}}{(4x^2 + 2x + 5) \cdot \frac{1}{x^2}} = \lim_{x \to \infty} \frac{2x^2 \cdot \frac{1}{x^2} + 2 \cdot \frac{1}{x^2} - 1 \cdot \frac{1}{x^2}}{4x^2 \cdot \frac{1}{x^2} + 2 \cdot \frac{1}{x^2} + 5 \cdot \frac{1}{x^2}} \)
\( = \lim_{x \to \infty} \frac{\frac{2}{x^2} + \frac{2}{x^2} - \frac{1}{x^2}}{\frac{4}{x^2} + \frac{2}{x^2} + \frac{5}{x^2}} = \frac{\frac{2}{4} + \frac{2}{4} - \frac{1}{4}}{\frac{4}{4} + \frac{2}{4} + \frac{5}{4}} = \frac{\frac{1}{2}}{\frac{11}{4}} = \frac{2}{11} \)

Page 112.
\( \lim_{x \to \infty} x^3 = \infty \)
\( \lim_{x \to \infty} \frac{x^2 + 2}{x^4} = \lim_{x \to \infty} \frac{\frac{x^2}{x^4} + \frac{2}{x^4}}{\frac{x^4}{x^4}} = \frac{0 + 0}{1} = 0 \)

Section 2.3 Laws of Limits
#19. \[ \lim_{x \to 0} \frac{\sqrt{4x+2}}{x} = 0.25 \]

Table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\frac{\sqrt{4x+2}}{x})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.24984</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.25016</td>
</tr>
<tr>
<td>0.001</td>
<td>0.24998</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.25002</td>
</tr>
</tbody>
</table>

\[
\lim_{x \to 0} \frac{\sqrt{4x+2}}{x} = \frac{x}{x} = \frac{1}{\sqrt{4+2}} = \frac{1}{4} = 0.25
\]

Limit Laws:

Premise: \(c\) is a constant, the limits \(\lim_{x \to a} f(x)\) \(\lim_{x \to a} g(x)\) exist.

#2. Check the premises:

a. Do the limits: \(\lim_{x \to 2} f(x)\) \(\lim_{x \to 2} g(x)\) exist

\[
\lim_{x \to 2} f(x) = 2 (f(2) = 1) \quad \lim_{x \to 2} g(x) = 0
\]

\[
\lim_{x \to 2} [f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 2 + 0 = 2
\]

b. Do the limits: \(\lim_{x \to 1} f(x)\) \(\lim_{x \to 1} g(x)\) exist

\[
\lim_{x \to 1} f(x) = 1 \quad \lim_{x \to 1} g(x) \text{ does not exist.}
\]

\[
\lim_{x \to 1} g(x) = 2 \neq \lim_{x \to 1} g(x) = 1
\]

Can not use \(\lim_{x \to 1} [f(x) + g(x)] = \lim_{x \to 1} f(x) + \lim_{x \to 1} g(x)\)

Note: \(f(x) = \sqrt{x+1}, \quad g(x) = \sqrt{x}\)

\[
\lim_{x \to \infty} [f(x) - g(x)] = \lim_{x \to \infty} \left[ \sqrt{x+1} - \sqrt{x} \right]
\]

\[
= \lim_{x \to \infty} \left( \sqrt{x+1} - \sqrt{x} \right) \cdot \frac{\left( \sqrt{x+1} + \sqrt{x} \right)}{\left( \sqrt{x+1} + \sqrt{x} \right)}
\]

\[
= \lim_{x \to \infty} \frac{\left( \sqrt{x+1} \right)^2 - \left( \sqrt{x} \right)^2}{\left( \sqrt{x+1} + \sqrt{x} \right)^2} = \lim_{x \to \infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0
\]

Page 104.

3. \(\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)\)

\[
\lim_{x \to a} c = c
\]

\[
\lim_{x \to a} x = a
\]
\[
\lim_{x \to a} x^n = a^n
\]

Page 110

Theorem 2. If \( f(x) \leq g(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and the limit of \( f \) and \( g \) both exist as \( x \to a \), then
\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)
\]

Premises: 1. \( f(x) \leq g(x) \)
2. \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist

Remark: \( f \) and \( g \) may/may not be defined at \( a \).

Conclusion: \( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \)

Remark: Is it true?

Premises: 1. \( f(x) < g(x) \)
2. \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist

Remark: \( f \) and \( g \) may/may not be defined at \( a \).

Conclusion: \( \lim_{x \to a} f(x) < \lim_{x \to a} g(x) \)

Counter example: \( f(x) = \frac{1}{x^2}, \ g(x) = \frac{1}{x}, \ x > 1 \)

It is True:

Premises: 1. \( f(x) < g(x) \)
2. \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist

Remark: \( f \) and \( g \) may/may not be defined at \( a \).

Conclusion: \( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \)

**Example** Show \( \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 \).

Proof:
- \( -1 \leq \sin \frac{1}{x} \leq 1 \)
- \( -1 \cdot x^2 \leq x^2 \sin \frac{1}{x} \leq 1 \cdot x^2 \)
- \( x^2 \leq x^2 \sin \frac{1}{x} \leq x^2 \)
- \( \lim_{x \to 0} (-x^2) = \lim_{x \to 0} (x^2) = 0 \)

By The Squeeze Theorem, \( \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 \).

\[ f(x) = x^2 \sin \frac{1}{x} \]