4.2
Page 290: Rolle's Theorem
Page 291: Mean Value Theorem.
Page 293:
If $f'(x) = 0$ for all $x$ in an interval $(a, b)$, then $f(x) = c$ on $(a, b)$.
Page 294: Corollary.

#2. $f(x) = x^3 - 3x^2 + 2x + 5$, $[0, 2]$

1. $f(x)$ is continuous on $[0, 2]$
2. $f(x)$ is differentiable in $(0, 2)$
3. $f(0) = 0^3 - 3 \cdot 0^2 + 2 \cdot 0 + 5 = 5$
   $f(2) = 2^3 - 3 \cdot 2^2 + 2 \cdot 2 + 5 = 5$

By Rolle's Thm, there exists $c \in (0, 2)$, such that

$f'(c) = 3c^2 - 6c + 2 = 0$

Solve for $c$

$3c^2 - 6c + 2 = 0$

$c = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \frac{6 \pm \sqrt{36 - 24}}{2 \cdot 3} = \frac{3 \pm \sqrt{6}}{3}$

$c_1 = \frac{3 + \sqrt{6}}{3} \in (0, 2)$
$c_2 = \frac{3 - \sqrt{6}}{3} \in (0, 2)$

$f(c_1) = \left(\frac{3 + \sqrt{6}}{3}\right)^3 - 3 \left(\frac{3 + \sqrt{6}}{3}\right)^2 + 2 \cdot \frac{3 + \sqrt{6}}{3} + 5$

$= 4.6151 \Rightarrow \left(\frac{3 + \sqrt{6}}{3}, \frac{3 + \sqrt{6}}{3}\right)$

$f\left(\frac{3 - \sqrt{6}}{3}\right) = \left(\frac{3 - \sqrt{6}}{3}\right)^3 - 3 \left(\frac{3 - \sqrt{6}}{3}\right)^2 + 2 \cdot \frac{3 - \sqrt{6}}{3} + 5$

$= 5.3849 \Rightarrow \left(\frac{3 - \sqrt{6}}{3}, \frac{3 - \sqrt{6}}{3}\right)$

$f(x) = x^3 - 3x^2 + 2x + 5$
#4. $f(x) = x\sqrt{x + 6}$  \([-6, 0]\)

(1) Continuity on \([-6, 0]\)
(2) Differentiability in \((-6, 0)\)
(3) $f(-6) = f(0)$

\[
f(-6) = -6\sqrt{-6 + 6} = 0 \quad f(0) = 0\sqrt{0 + 6} = 0
\]

\[
f'(x) = x'\sqrt{x + 6} + x\left(\sqrt{x + 6}\right)'
\]

\[
= \sqrt{x + 6} + x \cdot \frac{1}{2\sqrt{x+6}}
\]

\[
= \sqrt{x + 6} + x \cdot \frac{1}{2\sqrt{x+6}}
\]

\[
= \frac{2(x+6)}{2\sqrt{x+6}} + \frac{x}{2\sqrt{x+6}} = \frac{2x+12+x}{2\sqrt{x+6}} = \frac{3x+12}{2\sqrt{x+6}}
\]

In order to find all the $c$ such that $f'(c) = 0$,
set $f'(x) = \frac{3x+12}{2\sqrt{x+6}} = 0$  \((-6, 0) \Rightarrow 3x + 12 = 0 \Rightarrow x = -4$

\[
c = -4 \in (-6, 0)
\]

\[
f(-4) = -4\sqrt{-4 + 6} = -4\sqrt{2}
\]

#6. Let $f(x) = (x - 1)^{-2}$. Show that $f(0) = f(2)$ but there us no number $c \in (0, 2)$ such that $f'(c) = 0$. Why does this not contradict Rolle’s Theorem?

\[
f(x) = (x - 1)^{-2} = \frac{1}{(x-1)^2}
\]
Although,
\[ f(0) = (0 - 1)^{-2} = 1. \]
\[ f(2) = (2 - 1)^{-2} = 1. \]
\( f(x) \) is not defined at \( x = 1 \in [0, 2] \), \( f(x) \) is not continuous at \( x = 1 \), thus not continuous on \( [0, 2] \).
Rolle’s Thm does not apply.

#12.
\[ f(x) = x^3 + x - 1, \quad [0, 2] \]
\[ f(x) = x^3 + x - 1 \]

Check conditions:
(1) \( f(x) = x^3 + x - 1 \) is continuous on \([0, 2]\)
(2) \( f(x) = x^3 + x - 1 \) is differentiable in \((0, 2)\)
\[ f(2) = 2^3 + 2 - 1 = 9 \]
\[ f(0) = 0^3 + 0 - 1 = -1 \]
\[ \frac{f(b) - f(a)}{b-a} = \frac{9-(-1)}{2-0} = \frac{10}{2} = 5 \]
\[ f(x) = 3x^2 + 1 \]
Solve \( 3x^2 + 1 = 5 \) for \( x \).
\[ 3x^2 = 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm \sqrt{\frac{4}{3}} = \pm \frac{2}{\sqrt{3}} = \pm \frac{2}{3} \sqrt{3} \]
\[ c = \frac{2}{3} \sqrt{3} \in [0, 2] \]
\[ f\left( \frac{2}{3} \sqrt{3} \right) = \left( \frac{2}{3} \sqrt{3} \right)^3 + \frac{2}{3} \sqrt{3} - 1 = \frac{14}{9} \sqrt{3} - 1 \]
The tangent line equation at \( \left( \frac{2}{3} \sqrt{3}, \frac{14}{9} \sqrt{3} - 1 \right) \) is
\[ y - \left( \frac{14}{9} \sqrt{3} - 1 \right) = 5 \left( x - \frac{2}{3} \sqrt{3} \right) \]
\[ y = 5 \left( x - \frac{2}{3} \sqrt{3} \right) + \left( \frac{14}{9} \sqrt{3} - 1 \right) = 5x - \frac{16}{9} \sqrt{3} - 1 \]

\[ y = 5x - \frac{16}{9} \sqrt{3} - 1 \]

**#14.** \( f(x) = \frac{x}{x+2} \), \([1,4] \]

\[ f(x) = \frac{x}{x+2} \]

(1) \( f(x) = \frac{x}{x+2} \) is continuous on \([1,4] \)

(2) \( f(x) = \frac{x}{x+2} \) is differentiable on \((1,4) \)

\[ f'(x) = \frac{1}{(x+2)^2} \]

\[ f(4) = \frac{4}{4+2} = \frac{2}{3} \]

\[ f(1) = \frac{1}{1+2} = \frac{1}{3} \]

\[ \frac{2}{(x+2)^2} = \frac{f(4)-f(1)}{4-1} = \frac{\frac{2}{3}-\frac{1}{3}}{3} = \frac{1}{9} \]

\[ \frac{2}{(x+2)^2} = \frac{1}{9} \Rightarrow (x+2)^2 = 18 \Rightarrow x+2 = \pm \sqrt{18} = \pm 3 \sqrt{2} \]

\[ x = -2 \pm 3 \sqrt{2} \]

\[ x = -2 + 3 \sqrt{2} \in (1,4) \]

\[ x = -2 - 3 \sqrt{2} < 0 \not\in (1,4) \]

\[ f(-2+3\sqrt{2}) = \frac{-2+3\sqrt{2}}{-2+3\sqrt{2}+2} = \frac{-2+3\sqrt{2}}{3\sqrt{2}} = \frac{1}{6} \left(-2 + 3 \sqrt{2}\right) \sqrt{2} \]

The tangent line equation at \((-2+3\sqrt{2}, \frac{1}{6} \left(-2 + 3 \sqrt{2}\right) \sqrt{2})\) is

\[ y = \frac{1}{9} \left[ x - \left(-2 + 3 \sqrt{2}\right) \right] + \frac{1}{6} \left(-2 + 3 \sqrt{2}\right) \sqrt{2} \]

\[ = \frac{1}{9} x - \frac{-2+3\sqrt{2}}{9} + \frac{\left(-2+3\sqrt{2}\right)\sqrt{2}}{6} \]

\[ = \frac{1}{9} x - \frac{-2+3\sqrt{2}}{9} + \frac{-2\sqrt{2}+6}{6} \]

\[ = \frac{1}{9} x - \left(\frac{-2+3\sqrt{2}}{9}\right)^2 + \frac{18}{18} \left(-2\sqrt{2}+6\right)^2 \]

\[ = \frac{1}{9} x + \left(\frac{-2\sqrt{2}+6}{18}\right)^2 \]

\[ = \frac{1}{9} x + \left(-\frac{-4+6\sqrt{2}}{18}\right) \]

\[ = \frac{1}{9} x + \frac{-6\sqrt{5}+18-4+6\sqrt{2}}{18} \]

\[ = \frac{1}{9} x + \frac{-6\sqrt{5}+18+4-6\sqrt{2}}{18} \]
\[ y = \frac{1}{9}x + \frac{11 - 6\sqrt{2}}{9} \]

That is, \( y = \frac{1}{9}x + \frac{11 - 6\sqrt{2}}{9} \)

#16. Let \( f(x) = \frac{x+1}{x-1} \). Show that there is no value of \( c \) such that \( f(2) - f(0) = f'(c)(2 - 0) \). Why does this not contradict the Mean Value Theorem?

Interval of observation: \([0, 2]\). \( f(x) = \frac{x+1}{x-1} \) is not defined at one points in the interval \( x = 1 \). Mean Value Theorem does not apply.

\[ f(x) = \frac{x+1}{x-1} \]

However,
\[
\frac{df}{dx} = \frac{1 \cdot (x-1) - (x+1) \cdot 1}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}
\]

\[ f(2) = \frac{2+1}{2-1} = 3 \]

\[ f(0) = \frac{0+1}{0-1} = -1 \]

\[ f(2)-f(0) = \frac{-2}{(x-1)^2} (2-0) \Rightarrow 3 - (-1) = \frac{-2}{(x-1)^2} (2-0) \]

\[ 3 - (-1) = \frac{2}{(x-1)^2} (2-0) \Rightarrow 4 = \frac{-2}{(x-1)^2} \cdot 2 \]

\[ 2 = \frac{-2}{(x-1)^2} \Rightarrow 1 = \frac{-1}{(x-1)^2} \Rightarrow (x-1)^2 = -1 \]

No solution.

Mean Value Thm and Intermediate Value Thm.

#18. \( 2x - 1 - \sin x = 0 \)

What does it mean that the equation has a root?

\[ f(x) = 2x - 1 - \sin x \]

\[ f'(x) = 2 - \cos x > 0 \quad -1 \leq \cos x \leq 1 \]

\( f(x) \) is a strictly increasing function.

Many choices for \( a \), and \( b \) such that

\[ f(a) = 2a - 1 - \sin a < 0 \]

\[ f(b) = 2b - 1 - \sin b > 0 \]

Say,

\[ f(-1) = 2(-1) - 1 - \sin(-1) = -3 - \sin(-1) < 0 \]
\[ f(\pi) = 2\pi - 1 - \sin \pi = 2\pi - 1 > 0 \]

\([-1, \pi]\]

\(f(x)\) is continuous on the interval: \([-1, \pi]\),
\(f(-1) < 0, f(\pi) > 0\)

There exists a number \(c \in [-1, \pi]\), such that \(f(c) = 0\)
\(f(0) = 2(0) - 1 - \sin(0) = -1 < 0\)
\(f(\pi) = 2\pi - 1 - \sin \pi = 2\pi - 1 > 0\)

\([0, \pi]\]

\(f(x)\) is continuous on the interval: \([-1, \pi]\),
\(f(0) < 0 \quad f(\pi) > 0\)

There exists a number \(c \in [0, \pi]\), such that \(f(c) = 0\)
\(f(0) = 2(0) - 1 - \sin(0) = -1 < 0\)
\(f(\frac{\pi}{2}) = 2 \cdot \frac{\pi}{2} - 1 - \sin \frac{\pi}{2} = \pi - 1 - 1 = \pi - 2 > 0\)

There are only one root.

\#20. Show that the equation \(x^4 + 4x + c = 0\) has at most two real roots.
\(f(x) = x^4 + 4x + c \quad f'(x) = 4x^3 + 4\)

Let \(f'(x) = 4x^3 + 4 = 0 \Rightarrow x^3 = -1 \Rightarrow x = -1\)

If \(x = -1\), then \(f'(-1) = 0\)

If \(x > -1\), then \(f'(x) > 0\)

If \(x < -1\), then \(f'(x) < 0\)

Observe intervals:

\([-\infty, -1] \quad [0, +\infty]\]

\(f(-1) = (-1)^4 + 4(-1) + c = -3 + c = c - 3\)

\#19. \(x^3 - 15x + c = 0 \quad [-2, 2]\)

\(f(x) = x^3 - 15x + c\)
\(f'(x) = 3x^2 - 15 \quad \text{Set} \quad f'(x) = 3x^2 - 15 = 0 \Rightarrow x = \pm \sqrt{5}\)

\(x = \pm \sqrt{5} \notin [-2, 2]\)

Since there is no change of the sign for function \(f'(x) = 3x^2 - 15\) and \(f'(x) < 0\) for all the \(x \in [-2, 2]\), since
\(f'(\pm 2) = 3(\pm 2)^2 - 15 < 0\)

The function \(f(x) = x^3 - 15x + c\) decreases in \([-2, 2]\)
\[ f(-2) = (-2)^3 - 15(-2) + c = 22 + c = c + 22 \]
\[ f(2) = (2)^3 - 15(2) + c = -22 + c = c - 22 \]
If \(-22 < x < 22\), or \(|x|< 22\), \(f(-2) > 0\), \(f(2) < 0\), that is \(f(-2) \cdot f(2) < 0\), By Intermediate Value Theorem, there exists a number \(r \in [-2, 2]\), such that \(f(r) = 0\).
Say, \(c = 10, -21, 0, 12 \in [-22, 22]\)
\(f(x) = x^3 - 15x + c\)

If \(|x| > 22\), \(f(-2) > 0\), \(f(2) > 0\) or \(f(-2) \cdot f(2) > 0\),
\(f(-2) < 0\), \(f(2) < 0\). There is no roots in \([-2, 2]\).
\(f(x) = x^3 - 15x - 23\)

Therefore, there is at most one root in the interval \([-2, 2]\) for the equation: \(x^3 - 15x + c = 0\)

#21. Let the equation be: \(ax^3 + bx^2 + cx + d = 0\) \(a \neq 0\)
and the function be \(f(x) = ax^3 + bx^2 + cx + d\)
#23. If \(f(1) = 10\) and \(f'(x) \geq 2\) for \(1 \leq x \leq 4\), how small can \(f(4)\) possibly be?
If $f(x)$ is continuous on $[1,4]$, and differentiable in $(1,4)$. Then by the mean value theorem, there exists $c \in (1,4)$, such that

$$\frac{f(4)-f(1)}{4-1} = f'(c) \geq 2,$$

that is

$$\frac{f(4)-10}{3} \geq 2 \Rightarrow f(4) - 10 \geq 6 \Rightarrow f(4) \geq 16$$

The smallest value of $f(4)$ is 16.