2.5 The Product and Quotient Rules

So far we have two important rules, \((cf)' = cf'\) and \((f + g)' = f' + g'\). Put together we get the

**linearity rule**: \((af + bg)' = af' + bg'\)

There are four more very important rules we will need. Specifically, we want to know how to differentiate a product \(x \cdot \sin(x)\), a reciprocal \(1/\sin(x)\), a quotient \(x/\sin(x)\), and a power \(\sin^n(x)\).

**Warm-Up** Calculate the derivatives of \(f(x) = x^2\), \(g(x) = x^4\) and \(f(x)g(x) = x^6\).

**Conclusion**

\((f \cdot g)' \neq\)

**The product rule** Okay, so we know what it’s not. But is there a way to get the derivative of a product from the derivatives of the separate factors? Yes! Let’s say we want to get the derivative of \(u(x)v(x)\) or \(uv\) for short. Look at the average slope, \(\frac{\Delta uv}{\Delta x}\).

\[
\frac{\Delta uv}{\Delta x} = \frac{u(x + h)v(x + h) - u(x)v(x)}{h}
\]

We can illustrate the numerator with a diagram. You will have to fill in some of the picture in class.

\[
\begin{array}{c|c}
\hline
& \hline \\
\hline & & \hline \\
& & \hline \\
\hline
\end{array}
\]

This shows that

\[
\frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x)}{\Delta x} = \frac{u\Delta v + v\Delta u + \Delta u\Delta v}{\Delta x}
\]

\[
= u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}
\]
And in the limit
\[
\lim_{\Delta x \to 0} \frac{duv}{dx} = \lim_{\Delta x \to 0} \left( u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right) = u \frac{dv}{dx} + v \frac{du}{dx} + 0 \frac{dv}{dx}
\]
which gives us

**product rule** \( (f \cdot g)' = f \cdot g' + g \cdot f' \)

**Task 1.** Use the product rule to calculate the derivative of \( x \sin(x) \)

---

**Reciprocal rule** We get the reciprocal rule from the product rule. Starting with the product of reciprocals,

\[
u \cdot \frac{1}{u} = 1
\]

Then take the derivative of each side.

\[
\left( u \cdot \frac{1}{u} \right)' = (1)'
\]

\[
u \cdot \left( \frac{1}{u} \right)' + \left( \frac{1}{u} \right) \cdot u' = 0
\]

And isolate the \((1/u)'

\[
\left( \frac{1}{u} \right)' = -\frac{1}{u^2} u'
\]

**Task 2.** Use the reciprocal rule to calculate the derivative of \( \frac{1}{\sin(x)} \). Notice this is the derivative of \( \csc(x) \).
**Quotient rule**  From the product rule and the reciprocal rule comes the quotient rule.

\[ \left( \frac{u}{v} \right)' = \left( u \cdot \frac{1}{v} \right)' \]

\[ = u \cdot \left( \frac{1}{v} \right)' + \frac{1}{v} \cdot u' \]

\[ = u \cdot -\frac{1}{v^2} v' + \frac{1}{v} \cdot u' \]

\[ = \frac{-uv' + u'}{v^2} \]

\[ = \frac{u'v - uv'}{v^2} \]

**Task 3.** Use the quotient rule to calculate the derivative of \( \frac{x}{\sin(x)} \)

**Task 4.** Use the quotient rule to calculate the derivative of \( \frac{\sin(x)}{\cos(x)} \). Notice that this is the derivative of \( \tan(x) \).
**The power rule**  The power rule is established by mathematical induction. We wish to show that
\[(u^n(x))' = n \cdot u^{n-1}(x) \cdot u'(x),\] or \[(u^n)' = nu^{n-1}u',\] for short. We start with \(n = 1\).
\[(u)' = 1 \cdot u^0 \cdot u' = u'

This is trivial but necessary. Next we go from the \(n\)th power to the \((n+1)\)st power and apply the product rule
\[(u^{n+1})' = (u^n \cdot u)'
= u^n \cdot u' + u \cdot (u^n)'
= u^n \cdot u' + u \cdot n \cdot u^{n-1} \cdot u'
= u^n \cdot u' + n \cdot u^n \cdot u'
= (n + 1) \cdot u^n \cdot u'

This proves our power rule for positive integer powers.

**Task 5.** Use the power rule to calculate the derivative of \(\sin^2(x)\). Notice that this confirms our old square rule.

**Extended power rule**  It can be shown that the power rule will extend to all rational powers. For example \(x^{3/2}' = \frac{3}{2}x^{1/2}\).
Task 6. Find the derivatives of the following functions.

a) \[ \frac{1}{1 + x^2} + \frac{1}{1 - \sin(x)} \]

b) \[ x^{3/2} \sin(x) + (\sin(x))^{3/2} \]

c) \[ [u(x)]^3 [v(x)]^5 \]
Example 1. A box is growing. The length is $t$, the width is $\frac{1}{1+t}$ and the height is $\sin t$. What is the rate of change (the derivative) of the volume?

Task 7. A cylinder has radius $r = \frac{t^{3/2}}{1 + t^{3/2}}$ and height $h = \frac{t}{1 + t}$. Find the rate of change of the surface area. Hint: The lateral area is the area of the rectangle whose height is $h$ and whose length is the circumference of the top/bottom circle, $2\pi r$. Then put this together with the areas of the top and bottom, $2 \cdot \pi r^2$.

The rules so far

<table>
<thead>
<tr>
<th>Rule</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$(au + bv)' = au' + bv'$</td>
</tr>
<tr>
<td>Product</td>
<td>$(uv)' = u'v + uv'$</td>
</tr>
<tr>
<td>Reciprocal</td>
<td>$\left(\frac{1}{u}\right)' = -\frac{1}{u^2}u'$</td>
</tr>
<tr>
<td>Quotient</td>
<td>$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$</td>
</tr>
<tr>
<td>Power</td>
<td>$(u^n)' = nu^{n-1}u'$</td>
</tr>
</tbody>
</table>