2.6 Limits

We have been working with limits just by using an intuitive idea of limits. This conceptual understanding will carry us a while but eventually we need a more rigorous definition.

**Limits of sequences as** \( n \to \infty \)

Here are some examples.

\[ a_n = \frac{2n + 5}{n - 4} \]

Here, \( a_n \to \frac{2n}{n} = 2 \), as \( n \to \infty \), since the 5 and the 4 become insignificant as \( n \) gets very large.

\[ a_n = \sqrt{a_{n-1}} + 2 \]

This may not be so easy to guess. \( a_n \to 4 \), as \( n \to \infty \), regardless of the starting value. One way to see what happens is to use a calculator.

\[ a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \left(\frac{1}{2}\right)^n \]

This too may not be obvious. Again a calculator may help to see that \( a_n \to 1 \), as \( n \to \infty \).

\[ a_1 = 1.2, \ a_2 = 1.23, \ a_3 = 1.231, \ a_4 = 1.2315. \]

I don’t think that this is at all obvious, but it turns out that this sequence, regardless of how it continues, must converge. We cannot say what the limit is, but we do know that there is a limit.

**Convergence to zero.** What does it mean that \( a_n \to 0 \)? Certainly the terms of the sequence must get small. But that’s not enough. Suppose all the terms are smaller than some really small number; suppose all terms are smaller than \( 10^{-100} \)? This is true of the sequence \( a_n = 10^{-100} - \frac{1}{n} \). But this sequence converges to \( 10^{-100} \), not 0. Suppose each term has to be smaller than the previous term. This is true of the third example above, but that sequence converges to 1. So how do we guarantee that a sequence converges to 0?

Suppose that for any small number you can think of, the terms of the sequence eventually go below that number and stay below. If this is true then the sequence must converge to 0.

The idea is that no matter how the sequence starts, and no matter how small a number you choose, if the terms eventually become less than the small number you chose, then the sequence converges to 0.

**Example 1** Let \( a_n = \frac{100n + 2}{n^2} \). For what value of \( n \) will \( a_n \) be less than \( \frac{1}{100} \) ? less than \( \frac{1}{1000} \)?

After playing around with a calculator for a while I believe the answer to the first question is around \( n = 10,000 \). I don’t think the TI-84 can handle the second question. This sequence does converge to 0, but very slowly. \( a_{1000} \approx \frac{1}{10} \). The point is that no matter how small a number we choose, eventually the terms of this sequence will fall below the chosen number and stay below.

**Example 2** Consider sequence \( 10^{-1}, 10^0, 10^{-4}, 10^{-3}, 10^{-7}, \ldots \) (the pattern is up one power, then down 4). For what value of \( n \) will terms fall below \( 10^{-10} \)? Continue finding the terms: \( 10^{-1}, 10^0, 10^{-4}, 10^{-3}, 10^{-7}, 10^{-6}, 10^{-10}, 10^{-9}, 10^{-13}, \ldots \) So if \( n > 7 \), then \( a_n < 10^{-10} \).

A sequence \( a_n \) converges to 0 provided that it eventually falls below an arbitrarily small number.
Conventionally we use the Greek letter epsilon, $\epsilon$, to represent this "arbitrarily small number". So we can re-state the condition as $a_n \to 0$, provided that all the $a_n < \epsilon$ for all $n$ beyond some fixed number, call it $N$.

Or once more, $a_n \to 0$ provided that

\[
\text{for any positive } \epsilon \text{ there is some number } N \text{ so that } a_n < \epsilon \text{ for all } n > N
\]

**Example 3.** Let $a_n = \frac{n^2}{2^n}$. The sequence starts with $\frac{1}{2}$, $\frac{4}{4}$, $\frac{9}{8}$, $\frac{16}{16}$, $\frac{25}{32}$, $\frac{36}{64}$, …. If you only look at the first three terms or so you might think it cannot converge to zero. Even though the numerator $n^2$ continues to grow, eventually the denominator $2^n$ grows much faster. The sequence does converge to zero.

![Graph of a(n) vs n]

We can extend the definition to sequences with negative terms by making the condition on the absolute value. In other words, $a_n \to 0$ provided that

\[
\text{for any positive } \epsilon \text{ there is some number } N \text{ so that } |a_n| < \epsilon \text{ for all } n > N
\]

**Example 4.** Let $a_n = (-1)^n \frac{1}{n}$. This gives $-1$, $\frac{1}{2}$, $-\frac{1}{3}$, $\frac{1}{4}$, ….

![Graph of a(n) vs n]

**Convergence to a nonzero number.** A sequence $a_n$ converges to a number $L$ provided that the difference between $a_n$ and $L$ eventually falls below an arbitrarily small number. In other words, $a_n \to L$, provided that

\[
\text{for any } \epsilon > 0, \text{ there is some number } N \text{ so that } |a_n - L| < \epsilon \text{ for every } n > N.
\]
Example 5. Let $a_n = \frac{2n + 5}{n - 4}$. Given any small positive number $\epsilon$, eventually (after some number $N$), $|a_n - 2| < \epsilon$.

![Graph of the sequence $a(n)$](image)

Example 6. The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ converges to $e \approx 2.71$.

![Graph of the sequence $a(n)$](image)

Limit of a Function $f(x)$, as $x \to a$. Now we want to consider the general situation. Suppose that the inputs of a function are getting closer and closer to some number $a$, do the outputs of the function approach some number? Once again, our important example is $f(x) = \frac{\sin x}{x}$. We know that the limit is 1, as $x \to 0$. This means that when $x$ is near 0, $f(x)$ must be close to 1. The idea is that $x \to 0$ forces $f(x) \to 1$. But in order to get $f(x)$ within $\epsilon$ of 1, it may be necessary to make $x$ even closer to 0. The point is that $x$ must be sufficiently close to 0 in order to force $f(x)$ within $\epsilon$ of 1.

![Graph of the function $\frac{\sin x}{x}$](image)
Task 1  How close to 0 does \(x\) have to be in order that \(f(x) = \sin x\) is within 0.1 of 1? Make some guesses and use a calculator to check.

In general, the limit of \(f(x)\), as \(x\) approaches \(a\), is \(L\), meaning \(x \to a\) forces \(f(x) \to L\), provided that

\[
\text{for every } \varepsilon > 0, \text{ there must be some } \delta > 0, \text{ so that if } |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon
\]

Task 2. Let \(f(x) = \frac{2x^2 - 5x - 3}{x - 3}\). We can easily guess by looking at a graph that \(\lim_{x \to 3} f(x) = 7\). How close to 3 does \(x\) have to be in order that \(f(x)\) be within 0.01 of 7? In other words if \(\varepsilon = 0.01\), what is a sufficiently small \(\delta\)? First, sketch a graph. Second make some guesses, and then check.

Calculating limits  For our purposes, generally we need to find the limit, but we will not need to prove that the number we find is the limit. There are two powerful methods. One is to construct a table of values. The other is to construct a graph. Both can give misleading information, but often can lead to a correct answer.

Task 3. Use any method to find the following limits.

a) \(\lim_{t \to 2} \frac{t^2 + 3}{t + 2}\)

b) \(\lim_{t \to 2} \frac{t^2 + 3}{t - 2}\)

c) \(\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}\) (What does it represent?)

d) \(\lim_{t \to 0} \frac{3\tan t}{\sin t}\)

e) \(\lim_{t \to 4} \frac{t^2 - 16}{t - 4}\)

e) \(\lim_{t \to 2} \frac{t^2 + 3}{t - 2}\)