3.4 Graphs

We no longer spend much time creating graphs by hand as this is done faster and better with some technology. However, it is as important as ever to be able to glean information from a graph, to be able to read a graph and to understand the connection between the derivatives (first and second) of a function and the behavior of the function itself. This behavior, increasing, decreasing, increasing rate of growth or decreasing rate of growth, is most efficiently displayed in the graph of the function.

**Key Aspects of Graphs** Some involve precalculus and some involve calculus.

1. The domain of $f$ tells us where the graph exists.

2. The zeros and sign of $f(x)$ give the $x$-intercepts, and tell us where the graph is above, where it is below the $x$-axis.

3. The zeros and sign of $f'(x)$ give the stationary points, and tell us where the graph is increasing, where it is decreasing.

4. The zeros and sign of $f''(x)$ give the inflection points, and tell us where the graph is bending up, where it is bending down.

5. The end behavior of $f(x)$ as $x \to \infty$, as $x \to -\infty$. This may involve horizontal or slant asymptotes.

6. The points at which $f(x) \to \infty$ or $f(x) \to -\infty$. Such points would yield vertical asymptotes.

7. Other considerations are whether $f(x)$ is even or odd, whether it is periodic, whether it has jumps or has corners (where $f'$ jumps), and any points of discontinuity.

**Example 1.** Consider the function $f(x) = \frac{x^2}{1-x^2}$, with derivatives $f'(x) = \frac{2x}{(1-x^2)^2}$ and $f''(x) = \frac{2+6x^2}{(1-x^2)^3}$. Let’s start with the graph.
a) Start by analyzing $f(x)$ itself. What is the domain? Where is it positive? negative? What is the end behavior? What are the asymptotes?

b) Next we analyze the derivative, $f'(x) = \frac{2x}{(1-x^2)^3}$. What are the zeros? Where is it positive ($f$ increasing)? negative ($f$ decreasing)?

c) Lastly, we analyze the second derivative, $f''(x) = \frac{2 + 6x^2}{(1-x^2)^3} = \frac{2 + 6x^2}{(1-x)^3(1+x)^3}$. What are the zeros? Where is it positive ($f$ concave)? negative ($f$ convex)?
Task 1. Analyze the function and its derivatives as in Example 1. above.

\[ f(x) = \frac{x^3}{4 - x^2}, \text{ with } f'(x) = \frac{12x^2 - x^4}{(4 - x^2)^2} \text{ and } f''(x) = \frac{96x + 8x^3}{(4 - x^2)^3} \]

Hint: To find the end behavior, do the long division to get \( f(x) = \frac{x^3}{4 - x^2} = -x + \frac{4x}{4 - x^2} \). Then it is easier to see that \( \lim_{x \to \infty} \frac{x^3}{4 - x^2} = \lim_{x \to \infty} \left(-x + \frac{4x}{4 - x^2}\right) = -x \). Which means that \( y = -x \) is an asymptote.
Task 2. Now we will switch the problem around. We will start with the graph and analyze the function and its derivatives from the graph itself. Use the graph to estimate the answers.

a) What is the domain of $f$?

b) What is(are) the $x$-intercepts?

c) What is(are) the vertical asymptotes?

d) What is(are) the horizontal asymptotes?

e) What is (are) the stationary points?

f) On what interval(s) is $f'$ positive? On what interval(s) is it negative?

g) What is (are) the inflection points?

h) On what interval(s) is $f''$ positive? On what interval(s) is it negative?
Task 3. Use the given information about $f$ and analyze the derivatives to find the stationary points, where $f$ is increasing, where it is decreasing, the inflections points and where $f$ is bending up, where it is bending down.

a) $\text{dom } f = (-\infty, \infty); \quad f(0) = 0; \quad f(5) = 0; \quad f'(x) = \frac{5(x-2)}{3x^{1/3}}; \quad f''(x) = \frac{10(x+1)}{9x^{4/3}}$

b) $\text{dom } f = [0, 2\pi); \quad f\left(\frac{\pi}{2}\right) = 0; \quad f\left(\frac{3\pi}{2}\right) = 0; \quad f'(x) = -\frac{2\sin x + 1}{\sin x + 2}; \quad f''(x) = \frac{2\cos x(\sin x - 1)}{(\sin x + 2)^3}$

Homework: Read Section 3.4, pp. 114-116, and take notes. Complete the Read-through questions a-i. Work exercises #3-29, odd.