3.5 Parabolas, Ellipses and Hyperbolas

The most important curves in mathematics are lines (linear approximations), sines and cosines (oscillations), exponentials (growth and decay), parabolas, ellipses and hyperbolas (light bulbs, cell phones, global positioning systems, and comets, to name just a few applications).

Conic Sections The lines, parabolas, ellipses and hyperbolas are collectively called conic sections. Imagine two cones opening in opposite directions infinitely far.

Then imagine the double infinite cone sliced by a plane. If the plane slices the cone horizontally, the intersection is a circle. If the plane slices at an angle above horizontal, but less than parallel to the side of the cone, the intersection is an ellipse. If the plane slices the cone parallel to the side of the cone, the intersection is a parabola. If the plane slices the cone vertically the intersection is a hyperbola.

The Parabola \( y = ax^2 + bx + c \)
The important reflective properties of parabolas involve the point called the focus. All rays coming straight down are reflected to the focus. This is the basis for a flashlight reflector, a satellite dish, telescopes, etc. From any point of the parabola, the distance to the focus is equal to the distance to the line called the directrix.
Example 1 Let $y = -2x^2 - 4x + 5$.

a) The vertex. With algebra we can complete the square to get the vertex, $y = -2(x + 1)^2 + 7$. Then the vertex is $(-1, 7)$. Alternatively, with calculus, we know that the vertex is a stationary point, so solving $y' = -4x - 4 = 0$, gives $x = -1$, and then $y(-1) = -2(-1)^2 - 4(-1) + 5 = 7$, which gives the same $(-1, 7)$.

b) The focus. To get the focus, we use the formula $p = \frac{1}{4a}$, where $p$ is the displacement from the vertex to the focus. [This formula is derived from the fact that any point on the parabola is equidistant from focus and the directrix.] So for this example, $p = \frac{1}{4(-2)} = -\frac{1}{8}$. Therefore the focus is $(-1, 7 - \frac{1}{8}) = (-1, \frac{55}{8})$.

c) The directrix is then on the opposite side of the vertex as the focus. So the directrix is $y = 7 - \left(-\frac{1}{8}\right) = \frac{57}{8}$.

The graph with both the focus and directrix is below.

![Graph of the parabola](image)

The Ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Notice that a circle is the special case of an ellipse where $a = b =$ radius. Another way to think of an ellipse is a circle which is stretched horizontally by a factor of $a$, and stretched vertically by a factor of $b$.

![Ellipse diagram](image)

An ellipse has two foci, located along the major axis. In the picture above, the foci are $(\pm f, 0)$, where $f^2 = a^2 - b^2$. This formula is derived from the fundamental definition of an ellipse: from any point on the ellipse the sum of the distances to the two foci is constant.
Example 2. Consider the ellipse \( x^2 - 2x + 2y^2 + 12y = 6 \).

a) Complete the squares. 
\[
(x-1)^2 + 2(y+3)^2 = 25, \quad \left( \frac{x-1}{5} \right)^2 + \left( \frac{y+3}{5/\sqrt{2}} \right)^2 = 1.
\]
From this we see that the center is \((1, -3)\), that the horizontal stretch is \(a = 5\) and the vertical stretch is \(b = 5/\sqrt{2} \approx 3.5\). Since the horizontal axis is the major axis \((a > b)\), the foci lie on the horizontal axis. First, \(f^2 = (5)^2 - \left( \frac{5}{\sqrt{2}} \right)^2 = 25/2 \Rightarrow f = 5/\sqrt{2}\). Second, the coordinates of the foci are \(\left( 1 \pm 5/\sqrt{2}, -3 \right) \approx (4.5, -3), (-2.5, -3)\).

b) The above is confirmed with a graph.

The Hyperbola \( \left( \frac{x}{a} \right)^2 - \left( \frac{y}{b} \right)^2 = 1 \)

I think of hyperbolas as inverted ellipses. In this case, the vertices and the foci lie on the horizontal. The big difference between the ellipse and the hyperbola is that the foci of an ellipse lie inside, between the vertices, whereas the foci of the hyperbola lie outside the vertices. In this generic example, the center is \((0, 0)\), the vertices are \((\pm a, 0)\), and the foci are \((\pm f, 0)\), where \(f^2 = a^2 + b^2\). As in the previous cases of the parabola and the ellipse, this formula comes from the definition of a hyperbola. From any point on the hyperbola, the difference of the distances to the two foci is always the same. The other interesting aspect of hypbolas is the asymptotes. Slant asymptotes and horizontal asymptotes are determined by the end behavior, that is by looking at the limits of \(y\) as \(x \to \pm \infty\). Rewriting the above equation gives \(\frac{y}{b} = \pm \sqrt{\left( \frac{x}{a} \right)^2 - 1} \) or \(y = \pm \frac{b}{a} \sqrt{x^2 - a^2}\). From this we can see that as \(x \to \pm \infty, \pm \frac{b}{a} \sqrt{x^2 - a^2} \to \pm \frac{b}{a} \sqrt{x^2} = \pm \frac{b}{a} x\). In other words, as \(x \to \pm \infty\), the curve of the hyperbola looks more and more like the staright lines \(y = \pm \frac{b}{a} x\).
Example 3. Consider \(9y^2 - 18y - 4x^2 - 4x = 26\).

a) Complete the square.

\[
9y^2 - 18y - 4x^2 - 4x - 28 = 0, \quad 9(y^2 - 2y + 1) - 4(x^2 + x + \frac{1}{4}) = 26 + 9 + 1, \quad 9(y - 1)^2 - 4(x + \frac{1}{2})^2 = \left(\frac{y - 1}{2}\right)^2 - \left(\frac{x + 1/2}{3}\right)^2 = 1
\]

b) The center is \((-\frac{1}{2}, 1)\). Since the orientation is vertical, the vertices are \((-\frac{1}{2}, 1 \pm 2) = (-\frac{1}{2}, 3), (-\frac{1}{2}, -1)\). Also, \(f^2 = 4 + 9 \Rightarrow f = \sqrt{13} \approx 3.6\). So the foci are \((-\frac{1}{2}, 1 \pm \sqrt{13}) \approx (-\frac{1}{2}, 4.6), (-\frac{1}{2}, -2.6)\).

And the asymptotes are \(y = \pm \frac{2}{3}(x + \frac{1}{2}) + 1\).

c) And the graph.

Example 4. Consider the ellipse \(x^2 - 2x + 4y^2 + 24y - 28 = 0\). To calculate the tangent line at the point \((2, 1)\), we get the slope \(\frac{dy}{dx}\) at that point. Start by differentiating each side of the equation,

\[
(x^2 - 2x + 4y^2 + 24y - 28)' = (0)' \Rightarrow (x^2)' - (2x)' + (4y^2)' + (24y) - (28)' = 0 \Rightarrow 2x - 2 + 8yy' + 24y' = 0.
\]

And solve for \(y'\), \(y' = \frac{1 - x}{12 + 4y}\). At \((2, 1)\), \(y'(2, 1) = -\frac{1}{16}\). And the equation of the tangent is \(y = -\frac{1}{16}(x - 2) + 1\). This is illustrated in the graph.
Task 1. Identify any relevant features (center, vertices, foci, asymptotes) and sketch the graph.

a) \((x + 3)^2 + 9y^2 = 9\)

b) \(\left(\frac{x}{2}\right)^2 - (y - 1)^2 = 1\)

c) \(x^2 - 2x - y = 6\)
Task 2. Find the equation of the tangent line to the hyperbola $4x^2 + 4x - 9y^2 + 18y = 8$ at the point $(-2, 0)$. Then sketch both together.