6.3 Growth and Decay in Science and Economics

The derivative of \( y = e^{ct} \) is \( y' = ce^{ct} \), which means that \( y' = cy \). This one differential equation and some others that are closely related will provide models for a wide variety of real-world situations.

**Differential Equations**  
Equations such as \( \frac{dy}{dt} = t \) and \( \frac{dy}{dt} = y \) are equations involving the derivative of a function \( y(t) \). The solution of a differential equation is a function \( y(t) \). More generally, the solution set is a family of functions.

For \( \frac{dy}{dt} = t \) the solution set is the set of all antiderivative functions, \( y(t) = \frac{1}{2}t^2 + C \). Each value of \( C \) gives a different curve, but they all share the same slope. The solution curves of \( y' = t \) are parabolas. We represent the differential equation with a *direction field or slope field*. Each arrow show a tangent to a solution curve.

**Exponential Growth and Decay**  
For \( \frac{dy}{dt} = y \) the solution set is the set of all functions, \( y(t) = Ce^t \). Each value of \( C \) gives a different curve. The solution curves of \( y' = y \) are exponential functions.

Often when dealing with a real-world situation we are looking for a specific solution function. Perhaps we know the starting conditions or *initial condition* of some situation. Say we know that \( \frac{dy}{dt} = cy \) and \( y(0) = y_0 \) is given. Then we have what is called an *initial-value problem*, which now has a unique solution. The solution is \( y(t) = y_0e^{ct} \). The constant \( c \) is the rate of growth/decay.
Example 1. Exponential decay. Carbon-14 has a half life of 5568 years. Find the decay rate, $c$. We know $y(t) = y_0e^{ct}$, and that $y(5568) = \frac{1}{2}y_0$.

Then $y(5568) = \frac{1}{2}y_0 \Rightarrow y_0e^{c \cdot 5568} = \frac{1}{2}y_0 \Rightarrow 5568c = \ln\left(\frac{1}{2}\right) \Rightarrow c = \frac{-\ln(2)}{5568}$.

Example 2. The populations of New York and Los Angeles are growing at 1% and 1.4% per year, respectively. Starting with 8 million for NY and 6 million for LA, when will the populations be equal?

Task 1. The effect of advertising decays exponentially. If 40% remember a new product after three days, find $c$. How long will it take for only 20% to remember?
**Task 2.** Most drugs in the bloodstream decay exponentially, $y' = cy$. The half-life of nicotine is 2 hours. After a 10-hour flight what fraction remains?

**Growth or Decay with a Source** The purely exponential model, $y' = cy$, allows for growth only from the initial value $y_0$. You put a certain amount of money ($y_0$) into an account paying a certain interest, but you make no additional deposits or withdrawals. The modified model allows for a source or a sink. It requires an additional term, $s$.

$$y' = cy + s$$

**Example 3.** Suppose $s = 1000$, a constant source. This might represent a deposit of $1000 per year into an existing account whose original deposit was $y_0$. What are the solutions of $y' = 0.02y + 1000$? Can we guess a solution? What about $y = e^{.02t+1000}$? It’s derivative is $y' = 0.02e^{.02t+1000} = 0.02y \neq 0.02y + 1000$. What about $y = e^{.02t} + 1000t$? It’s derivative is $y' = 0.02e^{.02t} + 1000 \neq y + 1000$. What about $y = Ae^{.02t} - \frac{1000}{.02}$?

Suppose the initial deposit was $y_0 = 8000$. We then have $y = Ae^{.02t} - \frac{1000}{.02}$ with $y(0) = 8000$. This gives that $A - \frac{1000}{.02} = 8000$, and $A = 8000 + \frac{1000}{.02} = 58000$. The solution is $y = 58000e^{.02t} - 50000$. 
Example 4. Suppose you win a scholarship of $5000 and deposit it into an account paying 2% annual interest. You withdraw $800 per year to help pay for books. In this case \( y' = 0.02y - 800 \). What is the general solution? From the previous example we might guess that the form should be \( y = Ae^{0.02t} + B \). Can we make this work? Can we find constants \( A \) and \( B \) that makes this a solution? Two statements must be true:

\[(Ae^{0.02t} + B)' = 0.02(Ae^{0.03t} + B) - 200 \text{ and } y(0) = 5000.\]

a) Find a formula for \( y(t) \).

b) How long will the scholarship last?

Task 3. What deposit \( y_0 \), invested at 3% annual interest, will provide $1000 per year payment for 20 years? Hint: Solve \( y' = 0.03y - 1000 \) with \( y(0) = y_0 \).

Homework: Read Section 6.3, pp. 242-247, and take notes. Complete the Read-through questions, a-k. Work exercises, 1-7, odd, 11-25, odd.