6.4 Logarithms

In terms of applications, the exponentials are more useful than the logarithms. However, the logarithms provide an antiderivative for one of the simplest rational functions $\frac{1}{x}$. In other words, since $(\ln y)' = \frac{1}{y}$,

$$\int_{y=1}^{x} \frac{1}{y} \, dy = \ln x$$

This is one way to define the logarithms. Either way, as a starting point or as a consequence, we have a means of calculating the area under $f(x) = \frac{1}{x}$.

Derivatives In combination with the chain rule, the logarithm provides a wealth of new functions whose derivatives we can now calculate.

Example 1. Calculate the derivatives.

a) $\frac{d}{dx}(\ln(x^3))$

b) $\frac{d}{dx}(\ln(\cos x))$

c) $\frac{d}{dx}(\ln(\ln x))$
Example 2. Calculate the derivatives by first taking the log of both sides and second the derivatives.

a) \( y = x^{\cos x} \)

b) \( y = (\ln x)^x \)

c) \( y = x^x \cdot \sqrt{x^2 + 1} \)

Task 1. Calculate the derivative of each function.

a) \( y = \frac{\ln x}{x} \)

b) \( y = \ln(\ln x^2) \)

c) \( y = x^{-1/x} \)
Task 2. Find the linear approximation for the given function at the indicated point.

a) \( f(x) = e^x \) at \( x = 0 \)

b) \( f(x) = \ln x \) at \( x = 1 \).

Task 3. Use the linear approximations you found in Task 2 to estimate \( e^{0.03} \).

Integrals Logarithms and substitution expand the functions whose antiderivatives we can determine.

Example 3. Calculate the integral \( \int_{t=2}^{5} \frac{t}{t^2 - 3} \, dt \).
Task 4. Calculate each integral.

a) \[ \int_{t=0}^{1} \frac{5}{3-2t} \, dt \]

b) \[ \int_{x=0}^{1} \frac{x^2}{x^3+1} \, dx \]

c) \[ \int_{t=2}^{e} \frac{dt}{t(ln(t))} \]

Task 5. Calculate the derivative of each function

a) \[ g(x) = \int_{t=x}^{x^2} \frac{1}{t} \, dt \]

b) \[ g(x) = \int_{t=x^3}^{1} \ln(t + t^2) \, dt \]
Example 4. We know that \( \int_{2}^{3} \frac{1}{x} \, dx = \ln 3 - \ln 2 = \ln(3/2) \) is the area under \( y = \frac{1}{x} \) from 1 to 3. But what is the "area under" \( y = \frac{1}{x} \) from -3 to -1? The answer cannot be \( \ln(-3) - \ln(-1) \) since neither term is defined. There are no such numbers. But clearly there is an area. Let’s try substitution, \( u = -x \).

Task 6. Calculate the area under \( f(x) = \frac{x^2}{x^3 - 1} \) from \(-\frac{1}{2}\) to \( \frac{1}{3} \). Hint: split the interval (and integral) into two, one negative and one positive.

Homework: Read Section 6.3, pp. 242-247, and take notes. Complete the Read-through questions, a-k. Work excerciess, 1-7, odd, 11-25, odd.