3. Right Triangle Trigonometry

3.1 Reference Angle
3.2 Radians and Degrees
3.3 Definition III: Circular Functions
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3.5 Velocities
3.1 Reference Angle

- Reference Angle
- Reference angle theorem
- Calculate trig function using reference angle
- Approximation
3.1 Reference Angle

- Definition. The reference angle for any angle $\theta$ in standard position is the **positive acute** angle between terminal side of $\theta$ and the $x$-axis. We denote the reference angle for $\theta$ by $\hat{\theta}$.
3.1 Reference Angle

Reference Angle Theorem

A trigonometric function of an angle and its reference angle differ at most in sign.

You need to remember the signs of trig function on each quadrant (Section 1.3, table 1)!
3.1 Reference Angle

Problems.

(1) Draw the given angle in standard position and name the reference angle.

(a) 253.8° [4] 
Ans. 73.8°

(b) −330° [10] 
Ans. 30°

(2) Find the exact value

(a) cos135° [14] 
− \frac{\sqrt{2}}{2}

(b) csc300° [24] 
− \frac{2}{\sqrt{3}}
3.1 Reference Angle

Problems.

(3) Use calculate to find
   (a) \( \cos 238° \) [30]  (b) \( \csc 93.2° \) [36]
   Ans. \(-\cos(58°) = -0.5299\)  Ans. \( \csc(3.2°) = 1.0016\)

(4) Find \( \theta \) to the nearest tenth of degree, \( 0° < \theta < 360° \),
   and if [50, 62]
   (a) \( \sin \theta = -0.3090 \) and \( \theta \in QIV \); (b) \( \sec \theta = -3.4159 \) and \( \theta \in QII \)
   Ans. 342.0°  Ans. 107.0°
3.1 Reference Angle

Problems.

(5) Find $\theta$, $0^{\circ} < \theta < 360^{\circ}$, and if

- $\sin \theta = -\frac{1}{\sqrt{2}}$ and $\theta \in QIII$ [68] Ans. $225^{\circ}$

- $\sec \theta = \sqrt{2}$ and $\theta \in QIV$ [78] Ans. $315^{\circ}$
3.2 Radians and Degrees

- Radian measure of an angle
- Conversion between radians and degrees
- Using calculator
3.2 Radians and Degrees

- Radian measure of an angle

In particular, if \( s = r \), then we get an central angle with exactly one radian.

\[
\theta \text{ (in radians)} = \frac{s}{r}
\]
3.2 Radians and Degrees

Radian and degree

- \[ 1 \text{ (rad)} = \left( \frac{180}{\pi} \right) \text{°} \]
- \[ 1° = \frac{\pi}{180} \text{ (rad)} \]
  where \( \pi \approx 3.1415926 \)

\[ \text{Multiply by } \frac{\pi}{180} \quad \text{Degrees} \]
\[ \text{Multiply by } \frac{180}{\pi} \quad \text{Radians} \]
3.2 Radians and Degrees

problems

(1) Let $\theta$ be a central angle in a circle of radius $r$ with arc length $s$. Find $\theta$ if $[2, 6]$

(a) $r = 10$ in, $s = 5$ in

(b) $r = \frac{1}{4}$ cm, $s = \frac{1}{8}$ cm

Ans. 0.5 radians

Ans. 0.5 radians

(2) For each given angle, $[14, 16]$

• Draw the angle in standard position
• Convert the angle to radian measure
• Name the reference angle to both degrees and radians

(a) $-120^\circ$

(b) $390^\circ$

Ans. ref = $60^\circ$, $\pi/3$ (rad)

Ans. Ref = $30^\circ$, $\pi/6$ (rad)
3.2 Radians and Degrees

problems

(3) Give the exact value of each of the following.
   (a) \( \cot \frac{\pi}{3} \) \([56]\)  
       \( \frac{1}{\sqrt{3}} \)
   (b) \( \sec \frac{5\pi}{6} \) \([60]\)
       \( \frac{2}{\sqrt{3}} \)

(4) For each given angle, \([44, 46]\)
   • Draw the angle in standard position
   • Convert the angle to degree measure
   • Name the reference angle to both degrees and radians
   (a) \( \frac{11\pi}{6} \)
       Ans. ref = 30°, \( \pi/6 \) (rad)
   (b) \( -\frac{5\pi}{3} \)
       Ans. Ref = 60°, \( \pi/3 \) (rad)
3.2 Radians and Degrees

problems

(5) Evaluate each expression using $x = \frac{\pi}{6}$. Provide exact answer. [74, 78]

(a) $\sin\left(x - \frac{\pi}{3}\right)$

(b) $-1 + \frac{3}{4} \cos\left(2x - \frac{\pi}{2}\right)$
3.3 Definition III: Circular Functions

- 3rd definition of trig functions
- Values of trig function at special angles
- Domain and range of trig functions
- Problems
3.3 Definition III: Circular Functions

If \((x, y)\) is any point on the unit circle, and \(t\) is the distance from \((1, 0)\) along the circumference of the unit circle, then the six Trigonometric Functions are defined as below.

<table>
<thead>
<tr>
<th>Trig Function</th>
<th>(circular function)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin t = y)</td>
<td></td>
</tr>
<tr>
<td>(\cos t = x)</td>
<td></td>
</tr>
<tr>
<td>(\tan t = \frac{y}{x})</td>
<td></td>
</tr>
<tr>
<td>(\cot t = \frac{x}{y})</td>
<td></td>
</tr>
<tr>
<td>(\sec t = \frac{1}{x})</td>
<td></td>
</tr>
<tr>
<td>(\csc t = \frac{1}{y})</td>
<td></td>
</tr>
</tbody>
</table>
3.3 Definition III: Circular Functions

\[
\left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \left( -\frac{\sqrt{3}}{2}, \frac{1}{2} \right) \quad \left( -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \quad \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right)
\]

\[
\left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \quad \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \quad \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)
\]

\[
(0, 1) \quad (0, -1) \quad (1, 0) \quad (-1, 0)
\]

\[
90^\circ \quad 120^\circ \quad 150^\circ \quad 180^\circ \quad 210^\circ \quad 225^\circ \quad 240^\circ \quad 270^\circ \quad 300^\circ \quad 315^\circ \quad 330^\circ \quad 360^\circ
\]

\[
(\pi/2, \pi/3) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2) \quad (\pi/2)
\]

\[
(0, \pi) \quad (\pi) \quad (2\pi) \quad (3\pi)
\]
### 3.3 Definition III: Circular Functions

Domain and range of trig functions

<table>
<thead>
<tr>
<th>Domain of the circular functions (trig function)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin t, \cos t )</td>
<td>all real numbers</td>
</tr>
<tr>
<td>( \tan t, \sec t )</td>
<td>all real numbers except ( t = \pi/2 + k\pi ) for any integer ( k )</td>
</tr>
<tr>
<td>( \cot t, \csc t )</td>
<td>all real numbers except ( t = k\pi ) for any integer ( k )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Range of the circular functions (trig function)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin t, \cos t )</td>
<td>([-1, 1])</td>
</tr>
<tr>
<td>( \tan t, \sec t )</td>
<td>all real numbers, or ((-\infty, \infty))</td>
</tr>
<tr>
<td>( \cot t, \csc t )</td>
<td>((-\infty, -1] \cup [1, \infty)), or all real numbers except ((-1, 1))</td>
</tr>
</tbody>
</table>
3.3 Definition III: Circular Functions

Problems

(1) Use the unit circle to evaluate each function.
   (a) $\cos(225^\circ)$ [2] \hspace{1cm} (b) $\cot\left(\frac{4\pi}{3}\right)$ [12]

\[
\cos(225^\circ) = -\frac{\sqrt{2}}{2} \hspace{1cm} \cot\left(\frac{4\pi}{3}\right) = \frac{1}{\sqrt{3}}
\]

(2) Using the unit circle, find the six trigonometric function for $\theta = \frac{5\pi}{3}$. [16]

\[
\sin(\theta) = -\frac{\sqrt{3}}{2} \hspace{1cm} \cos(\theta) = \frac{1}{2} \hspace{1cm} \tan(\theta) = -\sqrt{3}
\]

\[
csc(\theta) = -\frac{2}{\sqrt{3}} \hspace{1cm} \sec(\theta) = 2 \hspace{1cm} \cot(\theta) = -\frac{1}{\sqrt{3}}
\]
3.3 Definition III: Circular Functions

Problems

(3) Using the unit circle, find all $\theta$ between 0 and $2\pi$ and for which $\sin \theta = -\frac{1}{2}$. [22]

\[
\frac{7\pi}{6} \quad \text{and} \quad \frac{11\pi}{6}
\]

(4) If angle $\theta$ is in standard position and the terminal side of $\theta$ intersects the unit circle at the point $(-1/\sqrt{10}, -3/\sqrt{10})$, find $\sin \theta$, $\cos \theta$, and $\tan \theta$. [44]

$$\sin(\theta) = -\frac{3}{\sqrt{10}}, \quad \cos(\theta) = -\frac{1}{\sqrt{10}} \quad \text{and} \quad \tan \theta = 3$$
3.3 Definition III: Circular Functions

Problems

(5) Identify the argument of each function. \[54, 58\]
   (a) \( \sin(2A) \)  
   Ans. 2A
   (b) \( \cos(2x - \frac{3\pi}{2}) \)  
   ans. \( 2x - \frac{3\pi}{2} \)

(6) Use circular function definition to evaluate, \[68, 70\]
   (a) \( \cos(2\pi + \frac{\pi}{2}) \)  
   (b) \( \cos(2\pi + \frac{\pi}{3}) \)
3.3 Definition III: Circular Functions

Problems (related to domain and range)

(7) Determine if the statement is possible for some real number \( z \)? [74, 76, 80]

(a) \( \csc 0 = z \)  
(b) \( \sin 0 = z \)  
(c) \( \sin z = 1.2 \)

Ans. No  
Ans. Yes  
Ans. No
3.4 Applications

1) Formula for Arc Length (in a circle)

\[ s = r \theta \]

\( \theta \) must be in radians

2) Formula for Area of a Sector

\[ A = \frac{1}{2} r^2 \theta \]

Again, \( \theta \) must be in radians
3.4 Applications

(1) Find the arc length $s$ of a circle with central angle $\theta = 30^\circ$ and radius $r = 4$ mm. [8]

$$\frac{2\pi}{3} \text{ mm}$$

(2) Find the radius $r$ of a circle with central angle $\theta = \frac{3\pi}{4}$ and arc length $s = \pi$ cm. [34]

$$\frac{4}{3} \text{ cm}$$

(3) The minute hand of a clock is 1.2 cm long. How far does the tip of the minute hand travel in 40 minutes? [12]

$$1.6\pi \text{ cm}$$
3.4 Applications

(4) Find the area of the sector formed by central angle $\theta = 15^\circ$ and $r = 10 \text{ m}$. [46]

\[
\frac{25\pi}{6} \text{ m}^2
\]

(5) Example 6. If a sector formed by a central angle of $15^\circ$ has an area of $\pi/3$ square centimeters, find the radius of the circle.

\[2\sqrt{2} \text{ cm}\]
3.5 Velocities

Uniform Circular Motion

Linear velocity: \( \nu = \frac{s}{t} \)

Angular velocity: \( \omega = \frac{\theta}{t} \), \( \theta \) in radians

Relation between two velocities: \( \nu = r\omega \)
3.5 Velocities

(1) Find the linear velocity $\nu$ of a point moving with uniform circular motion if the point covers a distance $s = 12$ cm in time $t = 2$ seconds. \[4\]

$\nu = 6 \text{ cm/sec}$

(2) Find the distance $s$ covered by a point moving with linear velocity $\nu = 10$ feet per second for a time $= 4$ seconds. \[8\]

$s = 40 \text{ ft}$
3.5 Velocities

(3) Find the distance $s$ traveled by a point with angular velocity $\omega = 2$ radians per second on a circle of radius $r = 4$ inches for time $t = 5$ seconds. \[24\]

\[s = 40 \text{ in}\]

(4) Find the angular velocity $\omega$ associated with 20 rpm. \[30\]

\[\omega = 40 \text{ rad/min}\]