1. From a bin with 2 red pebbles, 5 green pebbles and 6 blue pebbles, how many must you take to be sure that you have
   a. at least 2 colors? (7)
   b. at least 3 colors? (12)
   c. at least 2 of the same color? (4)
   d. at least 4 of the same color? (9)

2. Prove that among five points select inside an equilateral triangle with side equal to 1, there always exists a pair at a distance not greater than \( \frac{1}{2} \).

   Solution: Partition the triangle into 4 small triangles with side \( \frac{1}{2} \), A, B, C, and D as follows. The PHP implies that two of our five points must fall in the the same area and simple geometry says those points have distance at most \( \frac{1}{2} \).

3. Show that at a math party with at least 2 people, there are 2 people who have shaken the same number of hands.

   Solution: The possible number of hand-shakes are \( \{0, 1, \cdots, n-1\} \). Notice that if somebody shakes hand \( n-1 \) times, nobody shakes hands 0 times and vice-versa. So we have \( n \) people (pigeons) and \( n-1 \) possible numbers (holes), and by the PHP two people must have the same number.

4. Suppose each point of the plane is colored red or blue. Show that some line segment has ends in the same color.

   Solution: Consider 3 distinct points on a line. The PHP implies that two of them must be in the same color which will give us the required segment.
The following are more advanced exercises:

6. Prove that, among any fifty points in the interior of a $7 \times 7$ unit-square, there are two points whose distance apart is at most $\sqrt{2}$.

Proof. Divide the $7 \times 7$ square into 49 small squares of size $1 \times 1$. The largest distance between any two points in a small square is $\sqrt{1^2 + 1^2} = \sqrt{2}$. By the Pigeonhole principle, if 50 points are placed into 49 squares, there is at least one small square containing at least two of the 50 points. Thus the distance between those two points is at most $\sqrt{2}$. 
7. Prove that in any party of \( n \) people, one can always find at least two people A and B such that the number of acquaintances of A is the same as the number of acquaintances of B. (No one is an acquaintance of him/herself.)

Proof. Let \( a_i \) be the number of acquaintances of the \( i \)th person. Then \( 0 \leq a_i \leq n-1 \) (there are \( n-1 \) other people). With \( a_1, \ldots, a_n \) being objects and \( \{0, 1, \ldots, n-1\} \) being boxes, there are the same number of boxes and objects; either there is one box with at least two objects or each box has exactly one object. In the latter case, the box \( 0 \) and box \( n-1 \) each has one object. Note that \( a_i = 0 \) means that the \( i \)th person is not acquainted to anyone at the party, while \( a_i = n-1 \) means that the \( j \)-th person is acquainted to everyone at the party, in particular to the \( i \)-th person. This is impossible as the \( i \)th person is not acquainted to anyone (in particular to the \( j \)-th person). Hence one of the boxes \( 0 \) and \( n-1 \) has to be empty and there is a box with at least two objects; thus, there are two people with the same number of acquaintances.

8. Show that if \( n+1 \) integers are chosen from the set \( \{1, 2, \ldots, 2n\} \), then there are always two which differ by \( 1 \).

Proof. Let \( a_1 < a_2 < \cdots < a_n < a_{n+1} \) be the \( n+1 \) chosen numbers which we list in an increasing order. Then \( a_{n+1} - a_1 \leq 2n-1 \).

If \( a_{i+1} - a_i \geq 2 \) for all \( i = 1, \ldots, n \), then
\[
a_{n+1} - a_1 = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n+1} - a_n) \geq n \cdot 2 = 2n
\]
(Telescopes)
which contradicts the fact that \( a_{n+1} - a_1 \leq 2n - 1 \). Hence \( a_{i+1} - a_i \geq 2 \) cannot be true for all \( i \); thus, \( a_{i+1} - a_i \leq 1 \) for at least one \( i \).

Another approach is to use the Pigeonhole principle. The above actually also proves the Pigeonhole principle. First, partition numbers \( \{1, \ldots, 2n\} \) into \( n \)-boxes with two numbers in each as follows: \( \{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\} \). If the total of \( n+1 \) numbers are chosen from these \( n \)-boxes, the Pigeonhole principle shows that at least two numbers coming from the same box. Then one gets two numbers that differ by \( 1 \).

9. Use the PHP to prove that the decimal expansion of a rational number \( \frac{m}{n} \) is eventually repeating.

Proof. First we assume that \( n \) is a positive integer. Apply the division algorithm to get \( m = qn + r_0 \) with \( 0 \leq r_0 \leq n-1 \). Here is the integer part of the rational number \( \frac{m}{n} \). To compute the tenth place digit \( a_1 \), one uses the division algorithm
\[
r_0 \cdot 10 = a_1 n + r_1 \quad \text{with} \quad 0 \leq r_1 \leq n-1.
\]
More generally, to get the \( 10^{-i} \cdot f \) place digit, \( a_i \), one uses the remainder \( r_{i-1} \) and division algorithm
\[
r_{i-1} \cdot 10 = a_i n + r_i
\]
When \( i = n \), then the \( n+1 \) remainders \( r_1, r_2, \ldots, r_n \) have value ranging from \( 0 \) to \( n-1 \). Thus, by the pigeonhole principle, there must be two equal remainders, say, \( r_i = r_j \) with \( i < j \). Let \( j \) be the smallest possible number such that \( r_i = r_j \) for some \( i < j \). Then \( a_{i+1} = a_{j+1} \) with \( r_{i+1} = r_{j+1} \) and, inductively, \( a_{i+k} = a_{j+k} \) and \( a_i = a_{j+i} = a_{i+2(j-i)} = \cdots \). This shows that the decimal is repeating with the repeating part \( a_1 a_{i+1} \cdots a_{j-1} \).

10. A student has 37 days to prepare for an examination. From past experience, she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day however), there is a succession of days during which she will have studied a total of exactly 13 hours.
Proof. Let \( g_i \) be the number of hours she will study on the \( i \)th day. Then \( g_i \geq 1 \) is an integer and \( \sum_{i=1}^{37} g_i \leq 60 \)

Let \( a_i = g_1 + g_2 + \cdots + g_i \) \((*)\)

be the total number of hours she studied during the first \( i \) days. Then

\[
1 \leq a_1 < a_2 < \cdots < a_{37} \leq 60 .
\]

and by adding 13 to all parts of the inequality, we get

\[
14 \leq a_1 + 13 < a_2 + 13 < \cdots < a_{37} + 13 \leq 60 + 13 = 73
\]

The list \( a_1, a_2, \cdots, a_{37}, a_1+13, a_2+13, \cdots, a_{37}+13 \) of 74 numbers all have values in the set \( \{1, 2, \cdots, 73\} \). Now the PHP implies that there are two numbers having equal value. Those two numbers cannot both be in the list \( \{a_1, \cdots, a_{37}\} \), nor can both be in the list \( \{a_1+13, \cdots, a_{37}+13\} \), since numbers within each of the two lists have distinct values. Thus one must be in the first list and the other has to be in the second list, i.e., \( a_i = a_j + 13 \) or \( a_i - a_j = 13 \) for some \( i \) and \( j \). It is necessary that \( i > j \)

Now \( a_i - a_j = (g_1 + g_2 + \cdots + g_i) - (g_1 + g_2 + \cdots + g_j) \) from \((*)\)

\[
= g_{j+1} + \cdots + g_i = 13
\]

(from the \( j+1 \)-st term out to the \( i \)-th term)

This (last box) states that there is a succession of days where during which she will have studied a total of exactly 13 hours.