Chapter 16: Vector Calculus

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Example. #40 on page 1045

Find the work done by the force field \( \mathbf{F}(x,y,z) = x \sin y \mathbf{i} + y \mathbf{j} \) on a particle that moves along the parabola \( y = x^2 \) from \((-1,1)\) to \((2,4)\).

If \( \mathbf{F} \) is a force-field, then the work done is found by integrating the dot product of the force and the displacement.

\[
\text{Work} = \int_{a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

The curve \( y = x^2 \) can be parameterized as \( x = t, \ y = t^2 \), so 
\( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j}, \) and \( \mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j}. \)

\( \mathbf{F} = t \sin t^2 \mathbf{i} + t^2 \mathbf{j} \)

\[
\text{Work} = \int_{-1}^{2} (t \sin t^2 + 2t^3) \, dt = \left( -\frac{1}{2} \cos t^2 + \frac{t^4}{2} \right)_{-1}^{2}
\]

\[
= (-1/2) \cos 4 + 8 - ( (-1/2) \cos 1 + (1/2))
\]

\[
= (-1/2)(\cos 4 - \cos 1) + (15/2) = \frac{1}{2} (15 - \cos 4 + \cos 1)
\]
A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions, how much work is done by the man against gravity in climbing to the top?

We want to integrate $\mathbf{F} \cdot d\mathbf{r}$.

$\mathbf{F} = 185 \mathbf{k}$  \[\text{[Note, we are letting } +\mathbf{k} \text{ be "up"\]}

$\mathbf{r} = 20 \cos t \mathbf{i} + 20 \sin t \mathbf{j} + \frac{15t}{\pi} \mathbf{k}$  \[\text{where } 0 \leq t \leq 6\pi\]

$d\mathbf{r} = \langle -20\sin t, 20\cos t, (15/\pi) \rangle \ dt$

$\mathbf{F} \cdot d\mathbf{r} = \langle 0, 0, (185)(15)/\pi \rangle \ dt$

\[
\int_{0}^{6\pi} \frac{185(15)}{\pi} \ dt = 185(15)6 = 16650 \text{ ft-lb.}
\]

#44: Suppose there is a hole in the can of paint in Exercise 43 and 9 lb of paint leaks steadily out of the can during the man’s ascent. How much work is done?

This will change the force, but not the displacement.

$\mathbf{F} = \left(185 - \frac{9}{6\pi} t\right) \mathbf{k}$

\[
\int_{0}^{6\pi} \left(\frac{185(15)}{\pi} - \frac{15(9t)}{6\pi}\right) \ dt = \]

$= 16245 \text{ ft-lb}$
Example similar to #3 - #10 on page 1053

Determine whether or not \( \mathbf{F} \) is a conservative vector field. If it is, find a function \( f \) such that \( \mathbf{F} = \nabla f \).

\[ \mathbf{F}(x,y) = (x^2 + y)i + (y^2 + x)j \]

The domain of \( \mathbf{F} \) is \( \mathbb{R}^2 \) which is simply connected, so look at \( \frac{\partial P}{\partial y} \) and \( \frac{\partial Q}{\partial x} \).

\[ P = x^2 + y \quad Q = y^2 + x \]
\[ \frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 1 \]

So, yes, the field is a conservative vector field.

Now we must find \( f \) such that \( \nabla f = \mathbf{F} = (x^2 + y)i + (y^2 + x)j \)

\[ \frac{\partial f}{\partial x} = (x^2 + y) \quad \rightarrow \quad f(x,y) = \frac{x^3}{3} + xy + g(y) \]
\[ \frac{\partial f}{\partial y} = (y^2 + x) \quad \rightarrow \quad f(x,y) = \frac{y^3}{3} + xy + h(x) \]

For these two \( f(x,y) \) to match we must have \( g(y) = \frac{y^3}{3} \) and \( h(x) = \frac{x^3}{3} \), so

\[ f(x,y) = \frac{x^3}{3} + xy + \frac{y^3}{3} \]

Now it is your turn. Try this one: \( \mathbf{F}(x,y) = (y e^{xy} + 4x^3y)i + (x e^{xy} + x^4)j \)
Example. Similar to #12 – 18 on page 1054

a) Find a function $f$ such that $F = \nabla f$ and 

$$F(x,y,z) = 2xy^3z^4 \mathbf{i} + 3x^2y^2z^4 \mathbf{j} + 4x^2y^3z^3 \mathbf{k}$$

C: $x = t, y = t^2, z = t^3, \quad 0 \leq t \leq 2$

We know two ways to do this. We can do it the “old” way which we learned earlier, or we can do this as we were instructed, using the (2$^{nd}$) Fundamental Theorem for Line Integrals.

$$\nabla f(x,y,z) = F(x,y,z)$$

$$\frac{\partial f}{\partial x} = 2xy^3z^4 \quad \rightarrow \quad f(x,y,z) = x^2y^3z^4 + g(y,z)$$

$$\frac{\partial f}{\partial x} = 3x^2y^2z^4 \quad \rightarrow \quad f(x,y,z) = x^2y^3z^4 + h(x,z)$$

$$\frac{\partial f}{\partial x} = 4x^2y^3z^3 \quad \rightarrow \quad f(x,y,z) = x^2y^3z^4 + j(x,y)$$

So $g(y,z) = h(x,z) = j(x,y) = K$ and

$$\text{(a) } f(x,y,z) = x^2y^3z^4 + K$$

$$\int_0^2 \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(2)) - f(\mathbf{r}(0))$$

$\mathbf{r} = \langle x, y, z \rangle$, so $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, and thus $\mathbf{r}(2) = \langle 2, 4, 8 \rangle$ and $\mathbf{r}(0) = \mathbf{0}$

$$\text{(b) } \int_0^2 \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(2)) - f(\mathbf{r}(0)) = (2^2)(4^3)(8^4) = 2^{20}$$

Now here is the “old” way:

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\mathbf{F}(t) = \langle 2tt^{6}, 3t^{2}t^{4}t^{12}, 4t^{2}t^{6}t^{9} \rangle = \langle 2t^{19}, 3t^{18}, 4t^{17} \rangle$$

$$\mathbf{F} \cdot \mathbf{r}'(t) = 2t^{19} + 6t^{19} + 12t^{19} = 20t^{19}$$

$$\int_0^2 \mathbf{F} \cdot \mathbf{r}'(t) \ dt = t^{20} \bigg|_0^2 = 2^{20} - 0^{20} = 2^{20} \quad \text{which agrees.}$$