Example – page 837, problem # 46

Find the equations of the normal plane and osculating plane of the curve:

\[ x = t, \ y = t^2, \ z = t^3, \] at the point (1,1,1).

If we have a point on the plane and a normal to the plane, we can find the equation of the plane. Both the normal plane and the osculating plane contain the point (1,1,1).

A vector parallel to the tangent vector will be normal to the normal plane, and \( \mathbf{r}'(t) \) is parallel to \( \mathbf{T}(t) \).

\[ \mathbf{r}(t) = \langle t, t^2, t^3 \rangle \] so \( \mathbf{r}'(t) = \langle 1,2t,3t^2 \rangle \) and \( \mathbf{r}'(1) = \langle 1,2,3 \rangle \)

The equation of the normal plane is: \( x + 2y + 3z = 6 \).

To find the equation of the osculating plane we need a vector parallel to \( \mathbf{B}(1) \) to use as the vector normal to the plane. We can find such a vector by crossing vectors parallel to \( \mathbf{T}(1) \) and \( \mathbf{N}(1) \). [We don’t use \( \mathbf{T}(1) \) and \( \mathbf{N}(1) \) because they are unit vectors and contain ugly radicals. We are only concerned with their direction, so we can ignore the radicals (mostly).]

So, as above, a vector parallel to \( \mathbf{T}(1) \) is \( \mathbf{r}'(1) = \langle 1,2,3 \rangle \)

Finding a vector parallel to \( \mathbf{N}(1) \) is a bit more work, and no, there isn’t a shortcut.

\[ \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \] \[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \] \[ \mathbf{r}'(t) = \langle 1,2t,3t^2 \rangle \] \[ |\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4} \]

\[ \mathbf{T}(t) = \left( 9t^4 + 4t^2 + 1 \right)^{-1/2} \langle 1,2t, 3t^2 \rangle \]

After a lot of algebra and a bit of calculus, we get:

\[ \mathbf{T}'(t) = \left( 9t^4 + 4t^2 + 1 \right)^{-3/2} \langle -18t^3 - 4t, -18t^4 + 2, 12t^3 + 6t \rangle \]

\( \mathbf{T}'(1) \) is parallel to \( \mathbf{N}(1) \), and a vector parallel to \( \mathbf{T}'(1) \) is: \( \langle -11, -8, 9 \rangle \)

\[ \langle 1,2,3 \rangle \times \langle -11, -8, 9 \rangle = 14\langle 3, -3, 1 \rangle \], so let’s use \( \langle 3, -3, 1 \rangle \) as the vector normal to the osculating plane.

We have the normal vector \( \langle 3, -3, 1 \rangle \) and the point (1,1,1), so

The equation of the osculating plane is: \( 3x - 3y + z = 1 \).
Things that students tend to forget in the heat of the moment:

\( \mathbf{T}(t) \) is parallel to \( \mathbf{r}'(t) \) and \( \mathbf{N}(t) \) is parallel to \( \mathbf{T}'(t) \), \textbf{BUT}

\( \mathbf{N}(t) \) and \( \mathbf{T}'(t) \) are NOT parallel to \( \mathbf{r}''(t) \).

Therefore, you cannot find \( \mathbf{T}'(t) \) by simply finding \( \mathbf{r}''(t) \) and normalizing it.

\[ \mathbf{T}'(a) \neq (\mathbf{T}(a))' \text{ (where } a \text{ is constant) since } (\mathbf{T}(a))' = 0. \text{ Thus} \]

\[ \mathbf{N}(1) \propto \mathbf{T}'(1) \neq [\mathbf{T}(1)]' = 0. \]

To find \( \mathbf{N}(a) \) (where \( a \) is a constant), you don’t have to find \( \mathbf{N}(t) \), but you DO have to find \( \mathbf{T}'(t) \) and then \( \mathbf{T}'(a) \) and then normalize \( \mathbf{T}'(a) \). That will save you a bit of work (compared to finding \( \mathbf{N}(t) \) and then \( \mathbf{N}(a) \)), but not much.